Distributed and robust network localization

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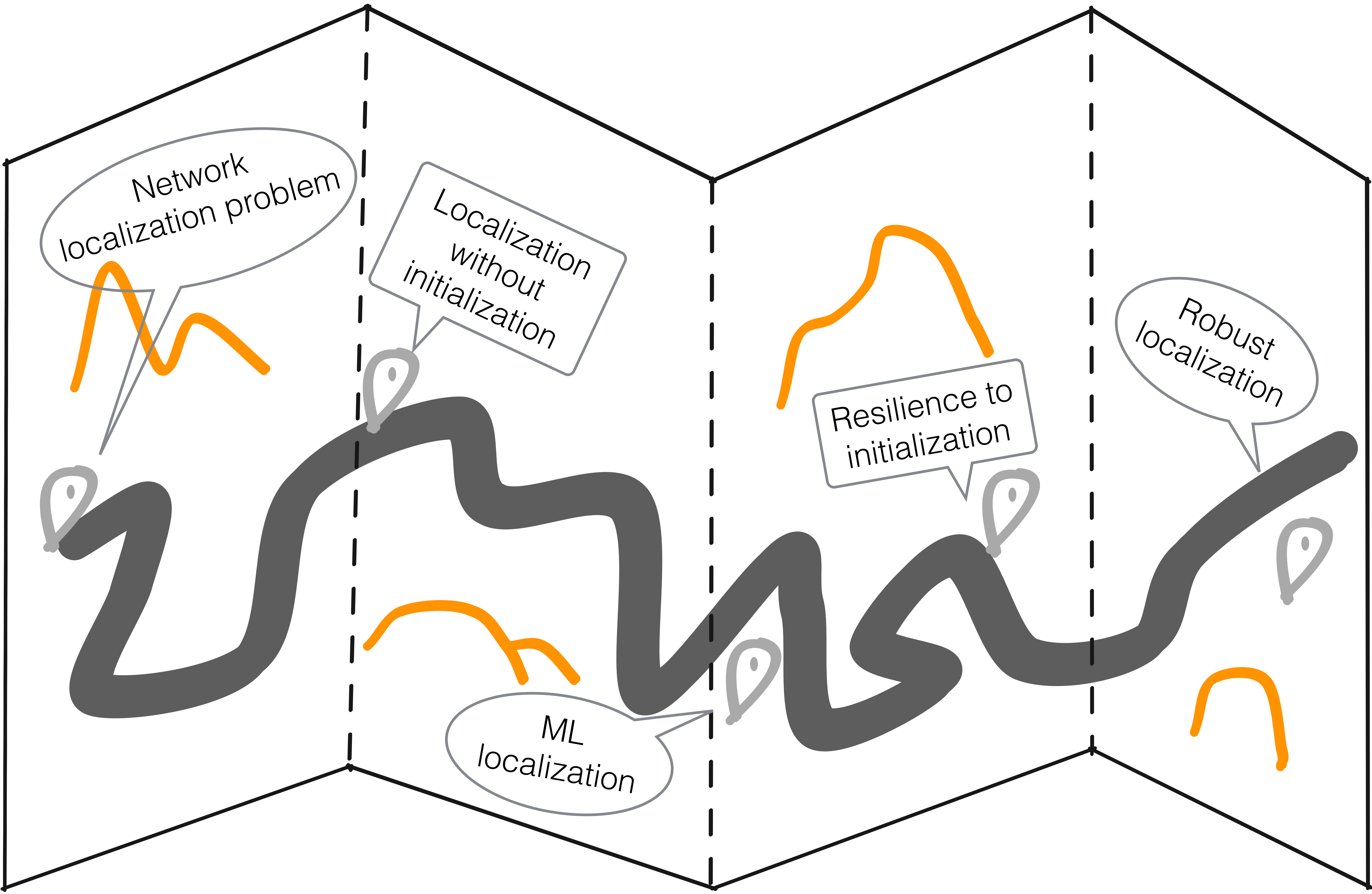
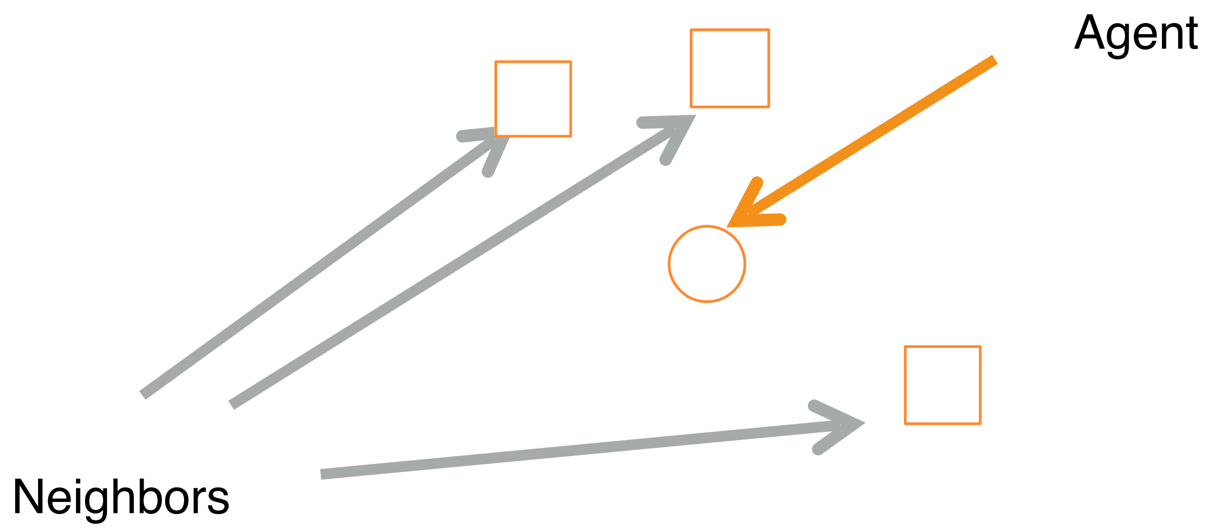
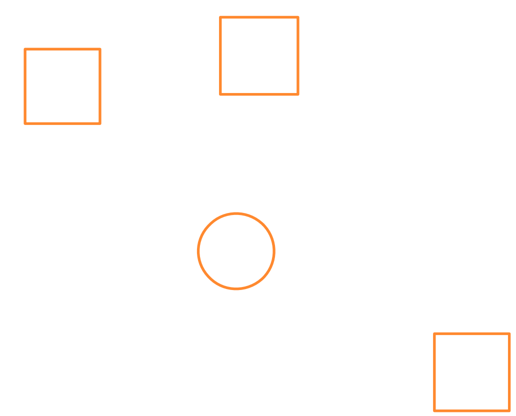
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR


Network localization problem
Localization without initialization
ML localization
Resilience to initialization
Robust localization

The problem



The problem



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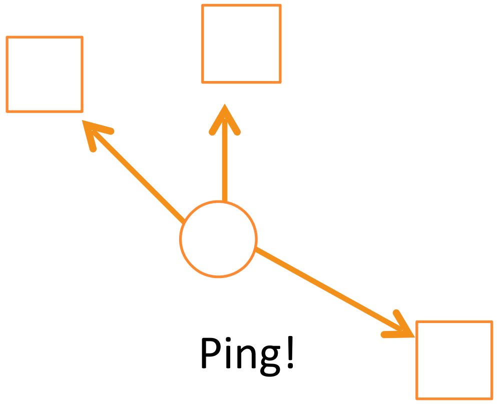
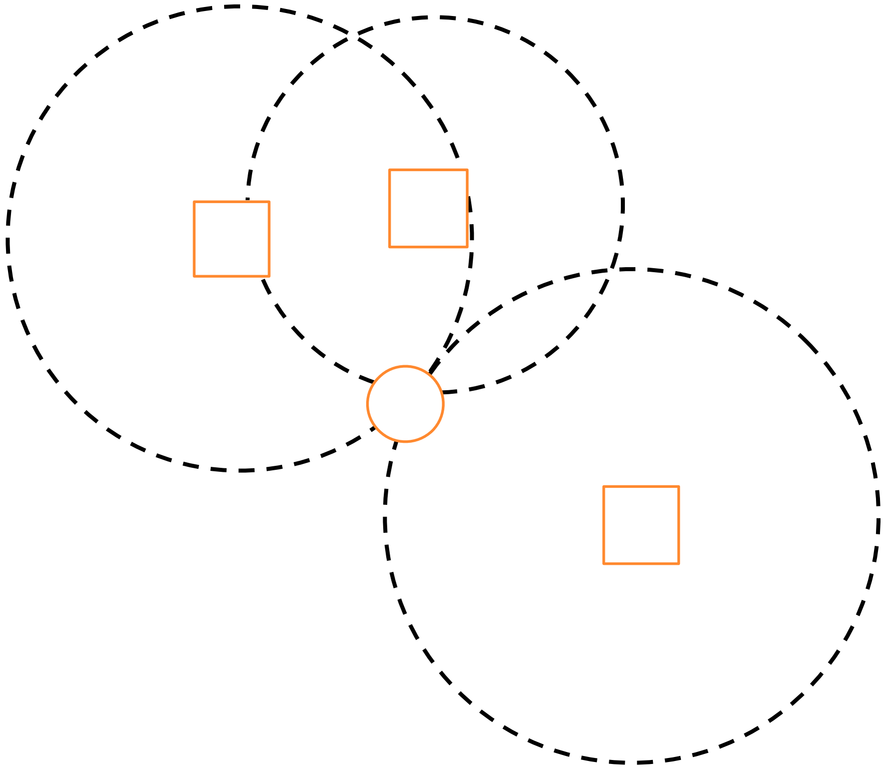
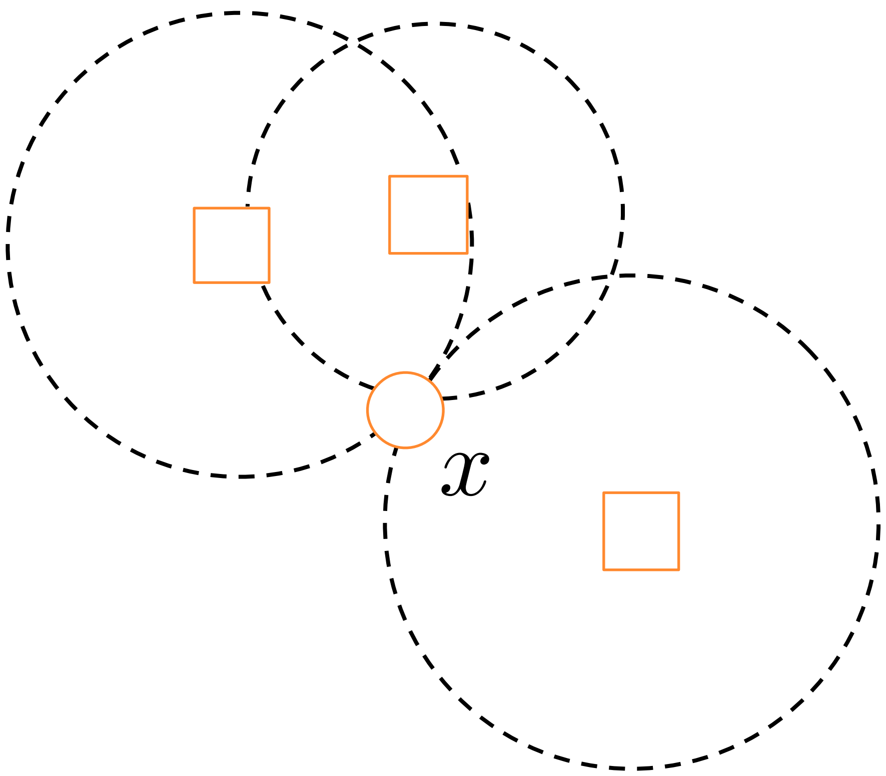


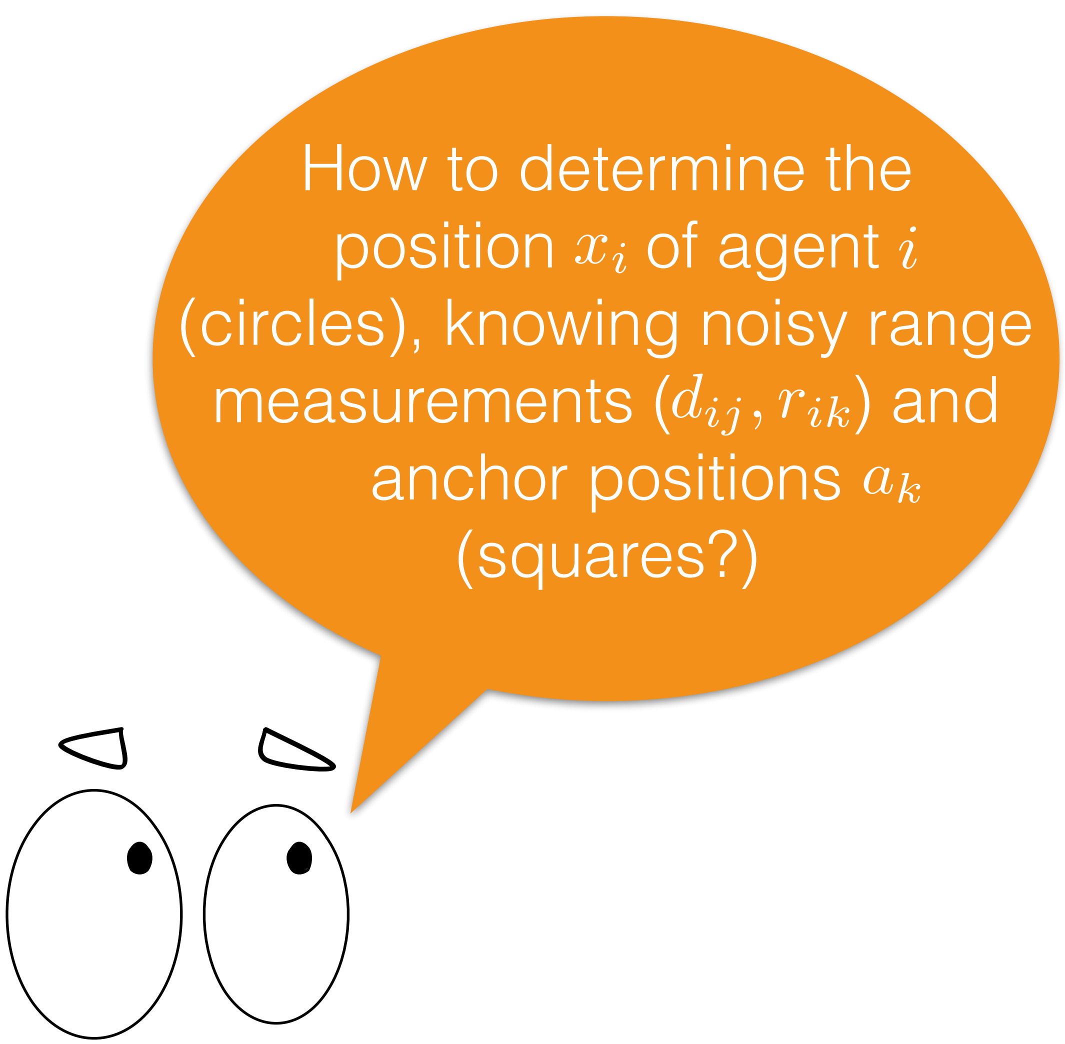
The problem

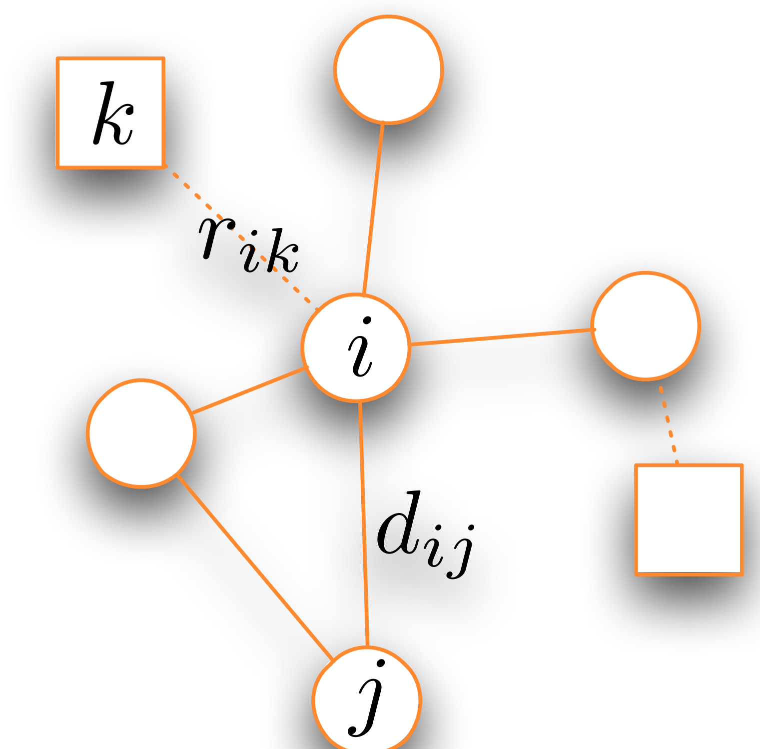


The problem

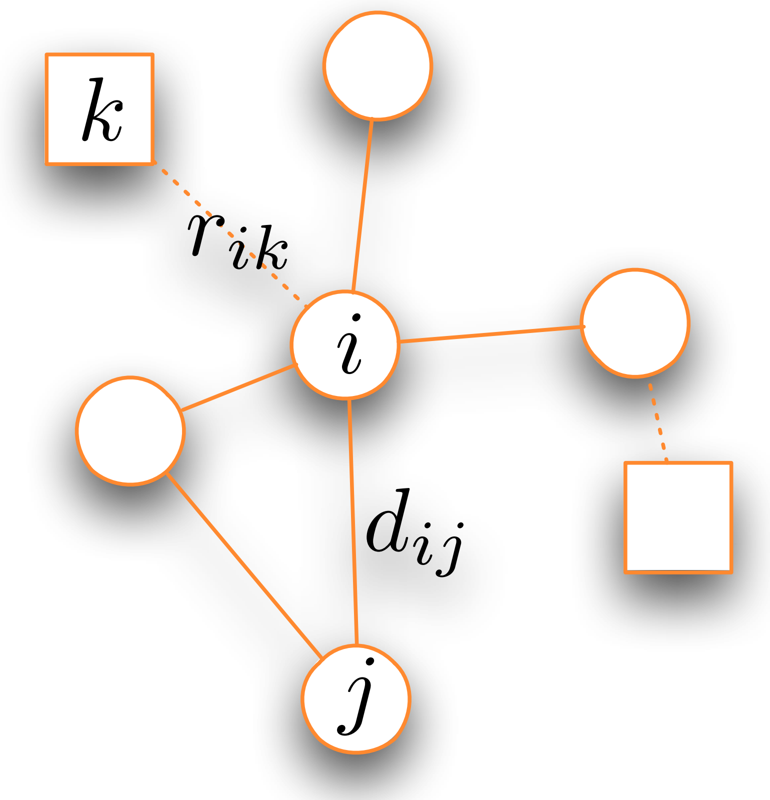


The problem


How to determine the position x_i of agent i (circles), knowing noisy range measurements (d_{ij}, r_{ik}) and anchor positions a_k (squares?)

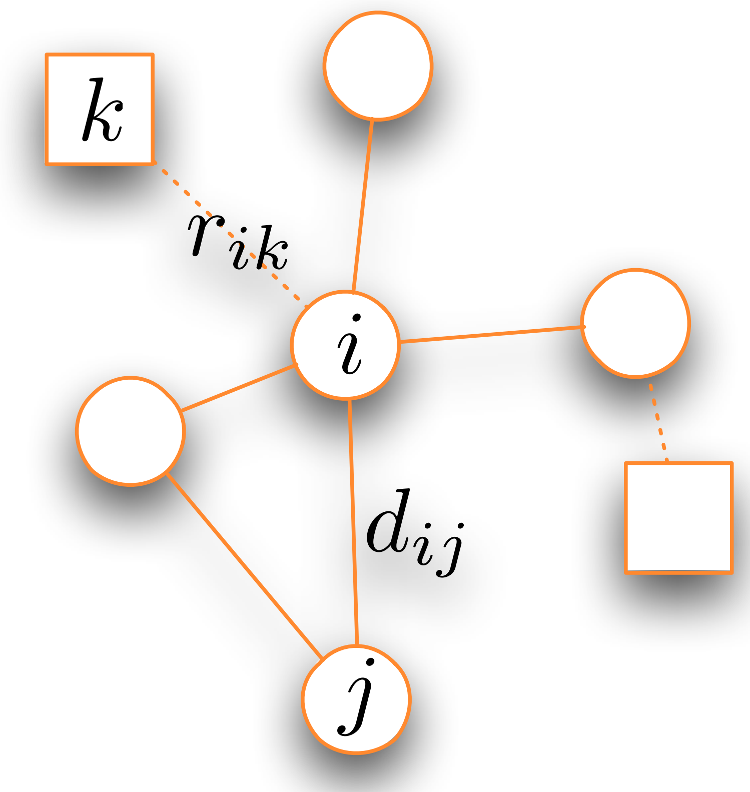

k
r_{ik}
i
d_{ij}
j

The problem



Measurements with added white Gaussian noise

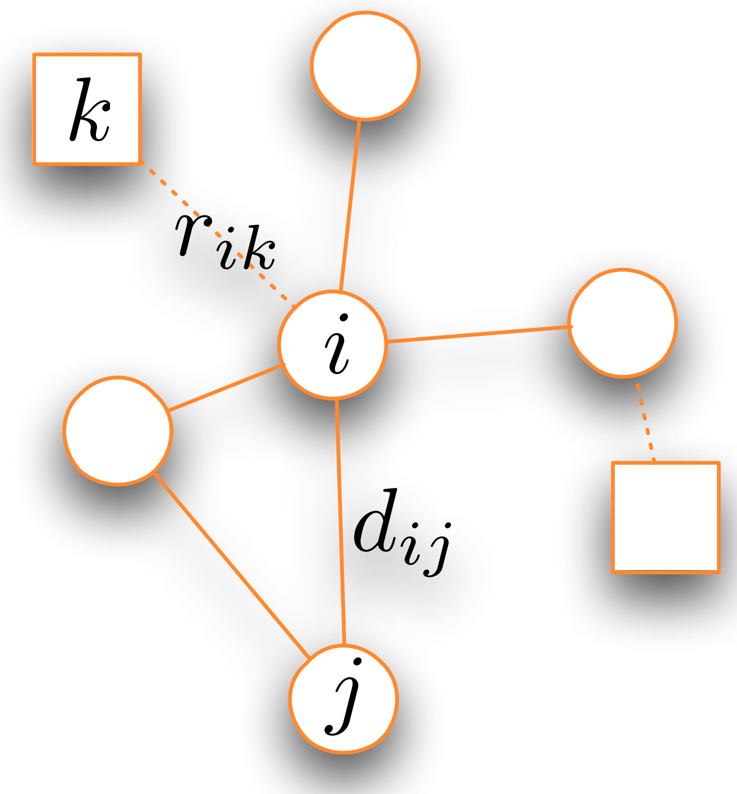
The problem



Measurements with added white Gaussian noise

$$r_{ik} = \|x_i^\star - a_k\| + \mathcal{N}(0, \sigma^2)$$

The problem



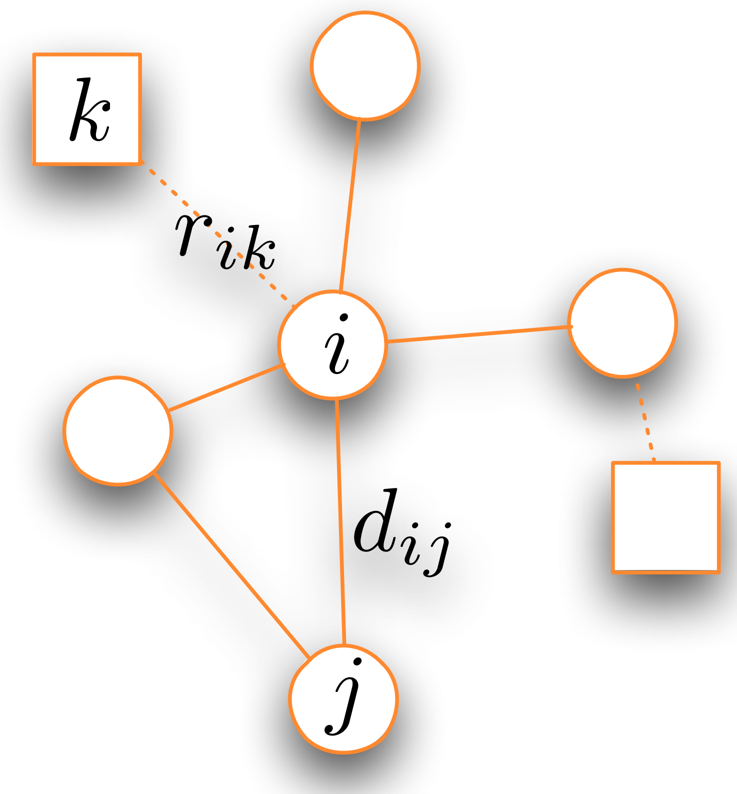
Measurements with added white Gaussian noise

↓

Maximum likelihood estimation

$$r_{ik} = \|x_i^\star - a_k\| + \mathcal{N}(0, \sigma^2)$$

The problem



Measurements with added white Gaussian noise

$$r_{ik} = \|x_i^\star - a_k\| + \mathcal{N}(0, \sigma^2)$$

↓

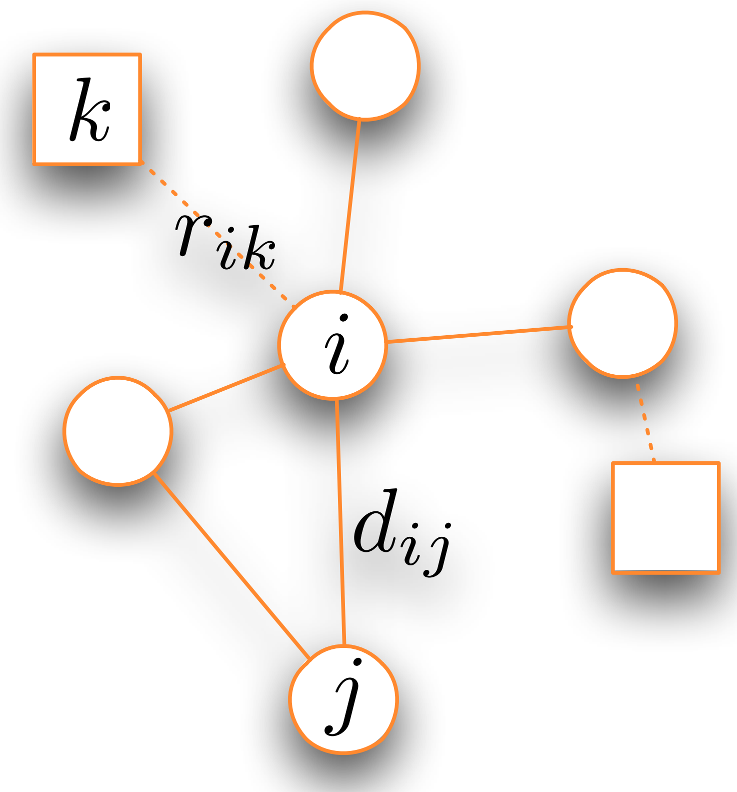
Maximum likelihood estimation

↓

Solve

$$\sum_{k \in \mathcal{A}_i} \frac{1}{2} \left(\|x_i - a_k\| - r_{ik}\right)^2$$

The problem



Measurements with added white Gaussian noise

$$r_{ik} = \|x_i^\star - a_k\| + \mathcal{N}(0, \sigma^2)$$

Maximum likelihood estimation

$$d_{ij} = \|x_i^\star - x_j^\star\| + \mathcal{N}(0, \sigma^2)$$

Solve

$$\underset{x}{\text{minimize}} \sum_{i \sim j} \frac{1}{2} \left(\|x_i - x_j\| - d_{ij} \right)^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \left(\|x_i - a_k\| - r_{ik} \right)^2$$

Simple and fast convex relaxation method

Under synchronous and asynchronous time models

Convex relaxation

$$\underset{x}{\text{minimize}} \sum_{i \sim j} \frac{1}{2} \left(\|x_i - x_j\| - d_{ij} \right)^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \left(\|x_i - a_k\| - r_{ik} \right)^2$$

Convex relaxation

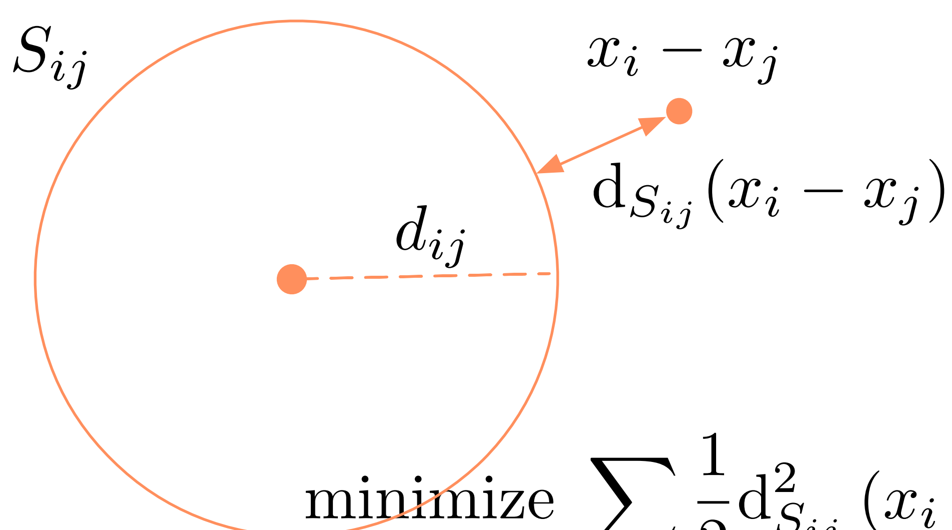
$$\underset{x}{\text{minimize}} \sum_{i \sim j} \frac{1}{2} \left(\|x_i - x_j\| - d_{ij} \right)^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \left(\|x_i - a_k\| - r_{ik} \right)^2$$

$$\underbrace{\left(\|x_i - x_j\| - d_{ij} \right)^2}_{\mathrm{d}^2_{S_{ij}}(x_i - x_j)} = \underset{y}{\text{minimize}} \; \|x_i - x_j - y\|^2$$

$$\text{subject to } \|y\| = d_{ij}$$

Convex relaxation

$$\underset{x}{\text{minimize}} \sum_{i\sim j} \frac{1}{2}\left(\|x_i - x_j\| - d_{ij}\right)^2 + \sum_i \sum_{k\in\mathcal{A}_i} \frac{1}{2}\left(\|x_i - a_k\| - r_{ik}\right)^2$$



$$\underset{x}{\text{minimize}} \sum_{i\sim j} \frac{1}{2}\mathrm{d}^2_{S_{ij}}(x_i - x_j) +$$

$$\sum_i \sum_{k\in\mathcal{A}_i} \frac{1}{2}\mathrm{d}^2_{S a_{ik}}(x_i)$$

Original problem

$$\underbrace{\left(\|x_i - x_j\| - d_{ij}\right)^2}_{\mathrm{d}^2_{S_{ij}}(x_i - x_j)} = \underset{y}{\text{minimize}} \|x_i - x_j - y\|^2$$

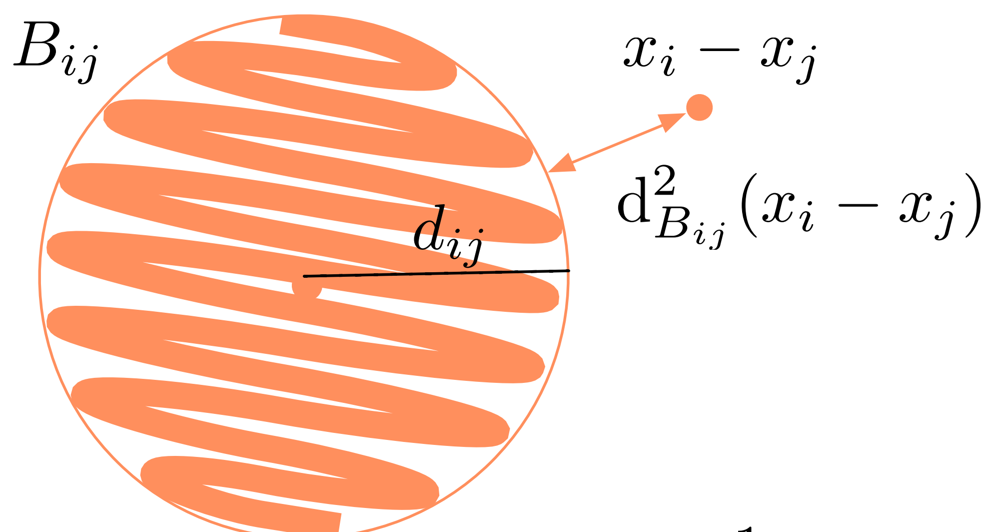
$$\text{subject to } \|y\| = d_{ij}$$

Convex relaxation

$$\begin{aligned} \mathrm{d}^2_{B_{ij}}(x_i - x_j) = \underset{y}{\text{minimize}}\ & \|x_i - x_j - y\|^2 \\ \text{subject to}\ & \|y\| \le d_{ij} \end{aligned}$$

Convex relaxation

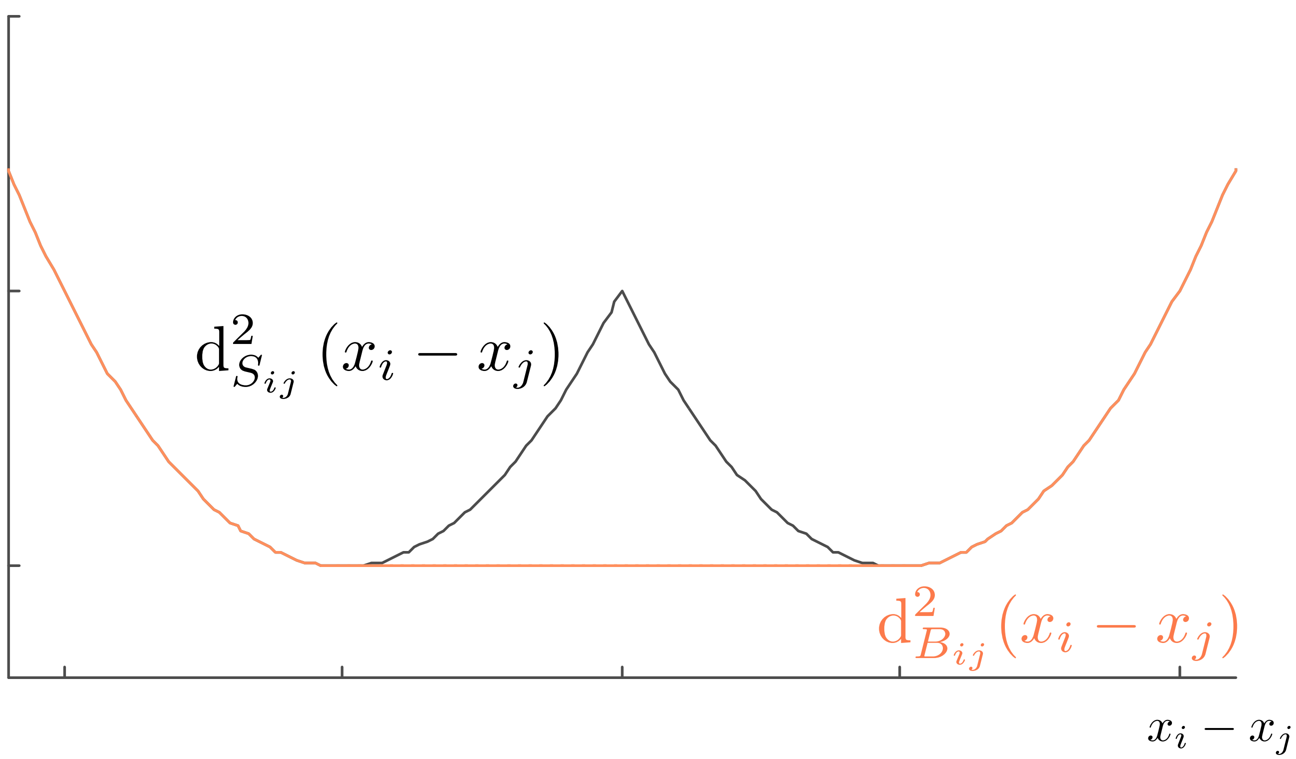
$$\mathrm{d}^2_{B_{ij}}(x_i - x_j) = \underset{y}{\text{minimize}}\ \|x_i - x_j - y\|^2$$
$$\text{subject to } \|y\| \leq d_{ij}$$



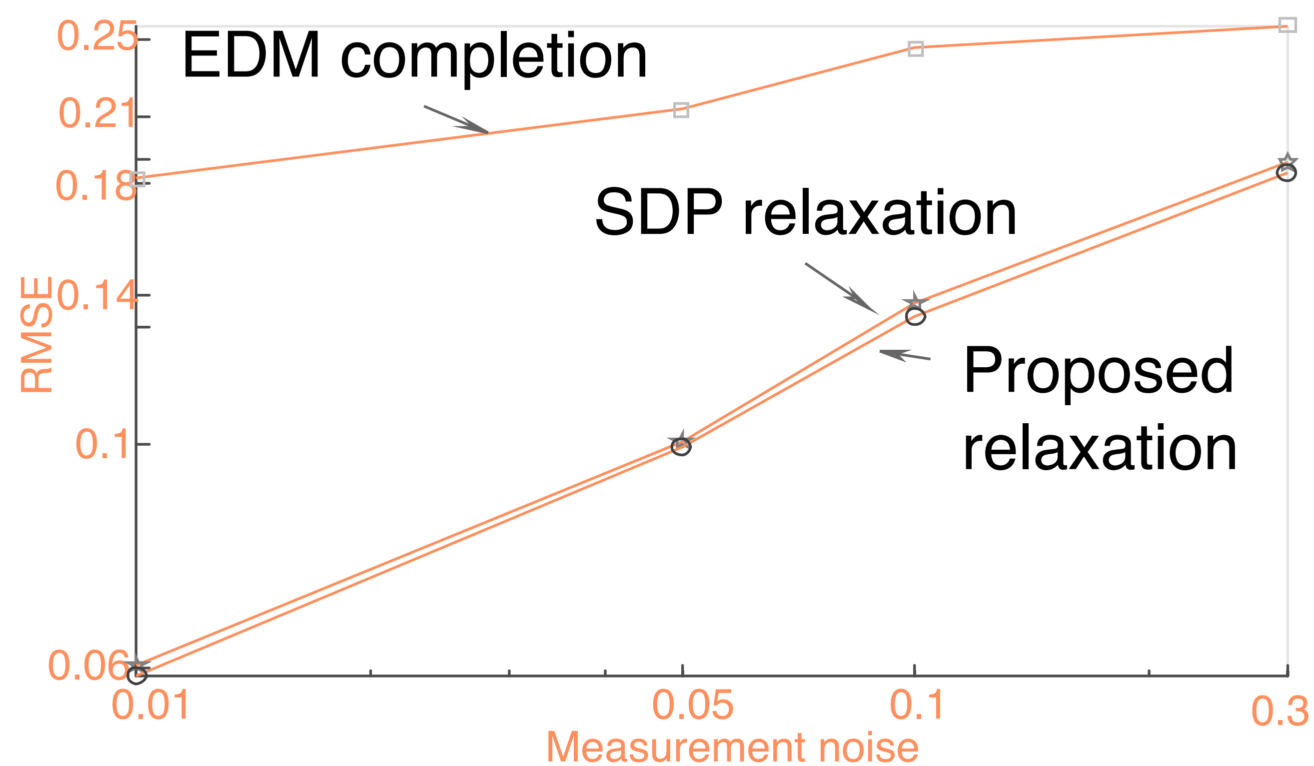
$$\underset{x}{\text{minimize}} \sum_{i \sim j} \frac{1}{2} \mathrm{d}^2_{B_{ij}}(x_i - x_j) +$$
$$\sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \mathrm{d}^2_{Ba_{ik}}(x_i)$$

Convex problem

Convex relaxation



Convex relaxation: experimental results



$$\mathrm{RMSE} = \sqrt{\frac{1}{n}\left(\frac{1}{M}\sum_{m=1}^{M}\|\hat{x}(m) - x^{\star}\|^2\right)}$$

Oguz-Ekim, *et al.*, TSP, 2011

Simonetto & Leus, TSP, 2014

Fast localization

It turns out that our function is:

Fast localization

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1. Convex;

Fast localization

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1. Convex;
2. Differentiable;

Fast localization

It turns out that our function is:

1. Convex;
2. Differentiable;
3. The gradient is Lipschitz continuous.

Fast localization

It turns out that our function is:

1. Convex;
2. Differentiable;
3. The gradient is Lipschitz continuous.

Gradient method with
optimal convergence rate

Distributed localization

The gradient is naturally distributed!

Distributed localization

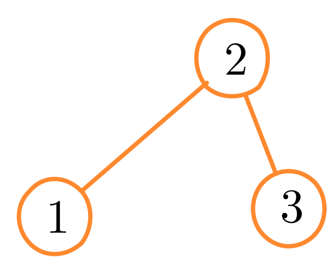
The gradient is naturally distributed!

Example

Distributed localization

The gradient is naturally distributed!

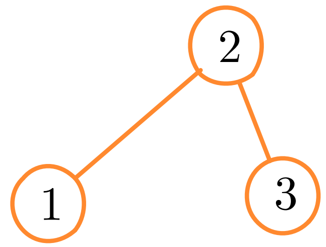
Example



Distributed localization

The gradient is naturally distributed!

Example

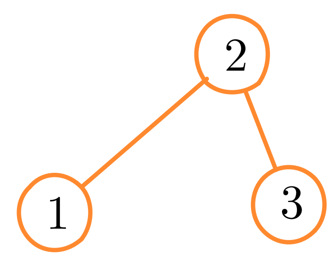


$$\nabla \hat{f}(x) = \begin{bmatrix} x_1 - x_2 + \mathrm{P}_{B_{12}}(x_1 - x_2) \\ 2x_2 - x_1 - x_3 - \mathrm{P}_{B_{12}}(x_1 - x_2) + \mathrm{P}_{B_{23}}(x_2 - x_3) \\ x_3 - x_2 - \mathrm{P}_{B_{23}}(x_2 - x_3) \end{bmatrix}$$

Distributed localization

The gradient is naturally distributed!

Example



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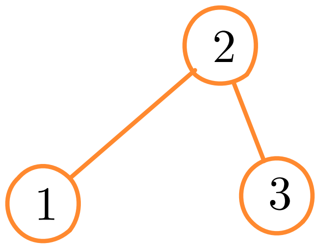
Distributed localization

The gradient is naturally distributed!

Example



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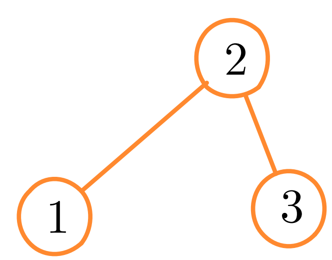
Distributed localization

The gradient is naturally distributed!

Example



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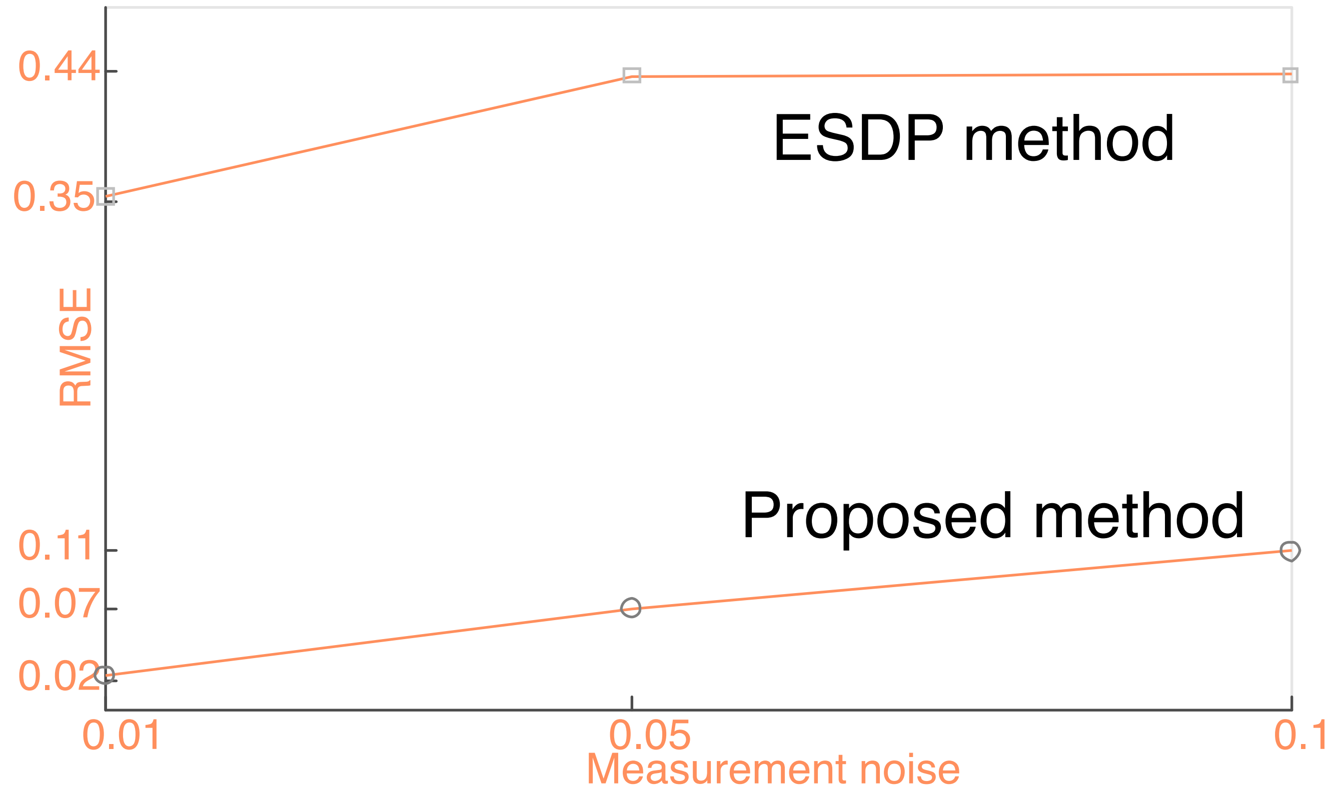
Experimental results



Number of communications per sensor

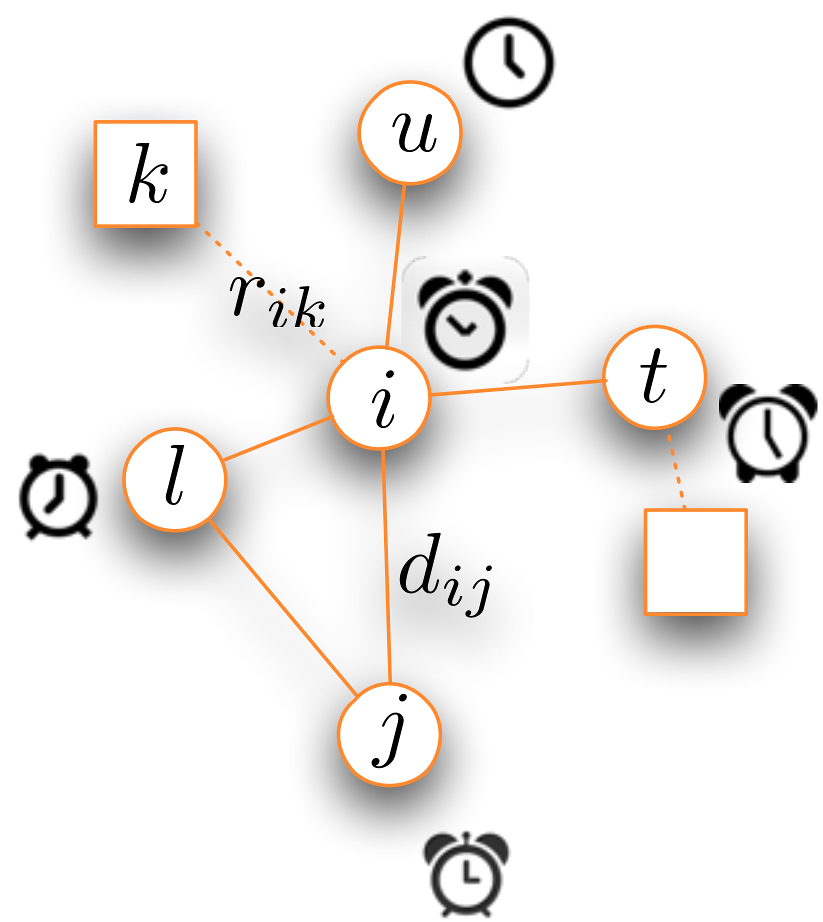
ESDP method	Proposed method
21600	2000

Simonetto & Leus, TSP, 2014

Asynchronous time model

$$T_i \sim PoissonProcess(\lambda)$$

$$\{Z_i^{k+1} - Z_i^k\} \sim Exp(\lambda)$$



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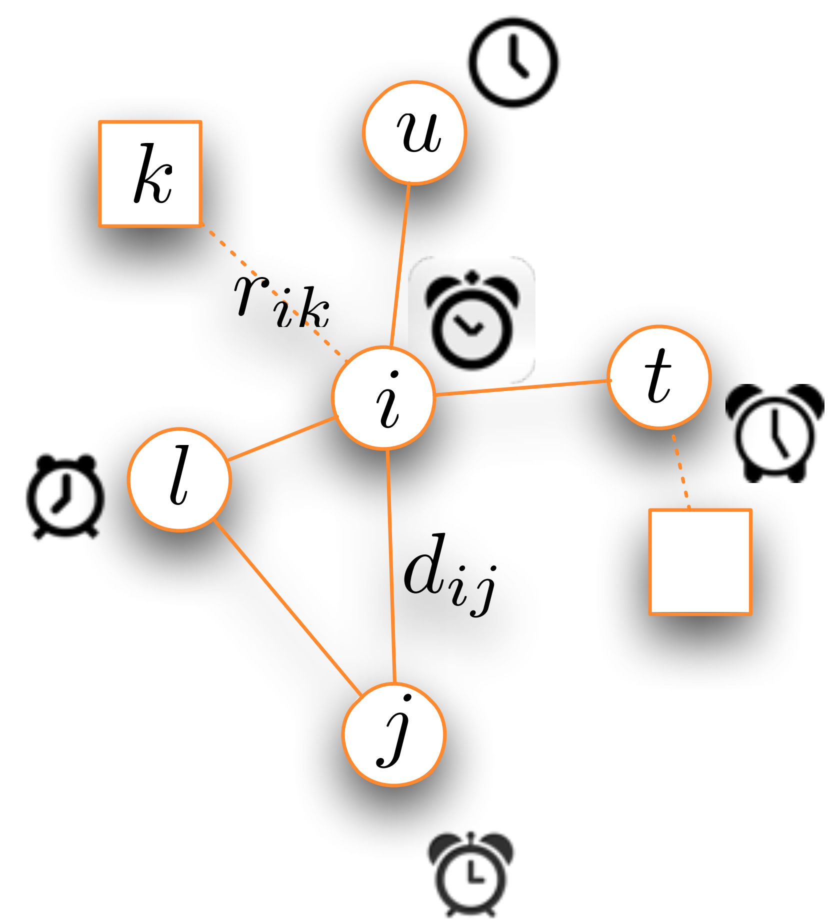
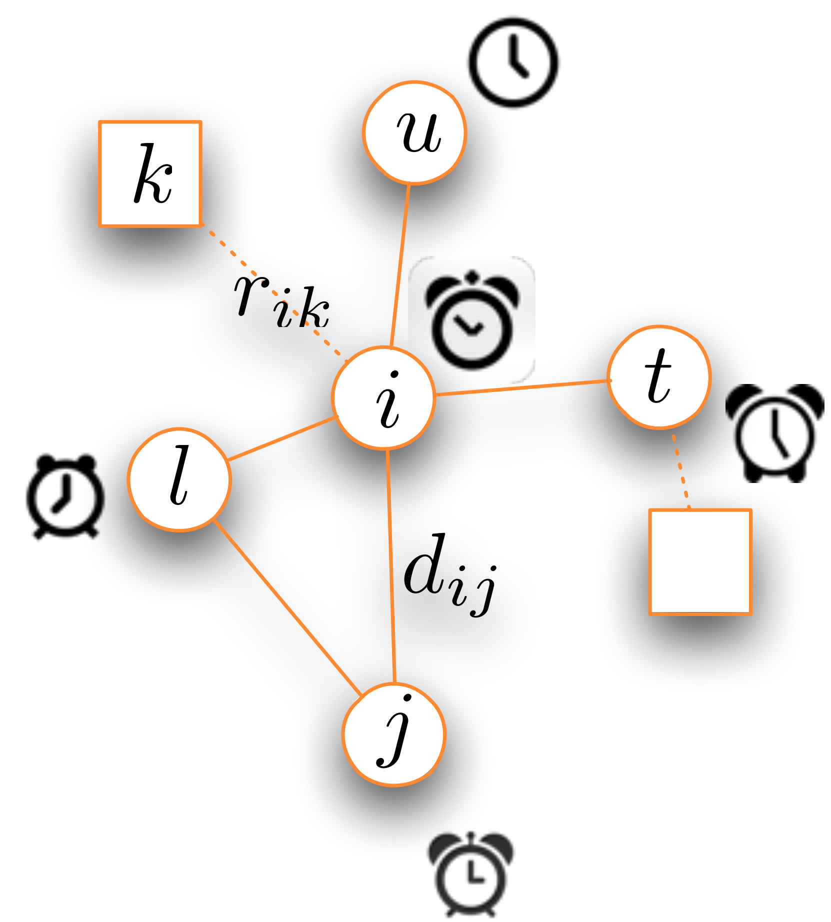


Asynchronous time model

$$T_i \sim PoissonProcess(\lambda)$$

$$\{Z_i^{k+1} - Z_i^k\} \sim Exp(\lambda)$$

$$\Updownarrow$$



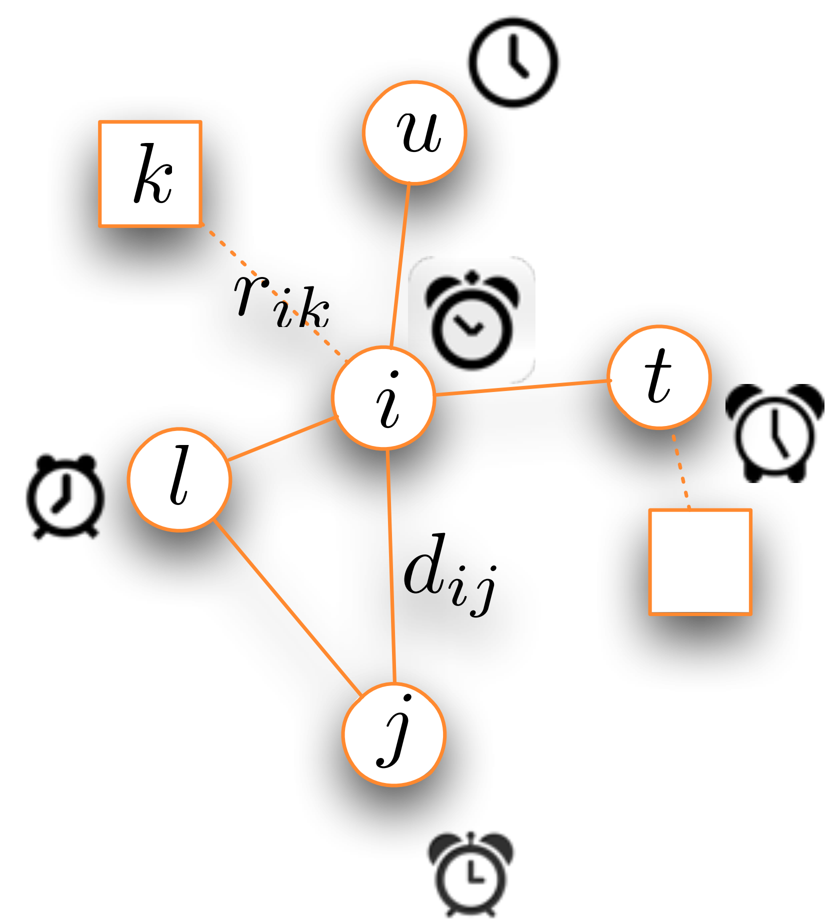
Asynchronous time model

$$T_i \sim PoissonProcess(\lambda)$$

$$\{Z_i^{k+1} - Z_i^k\} \sim Exp(\lambda)$$

$$\Updownarrow$$

$$T \sim PoissonProcess(N\lambda)$$



Asynchronous time model

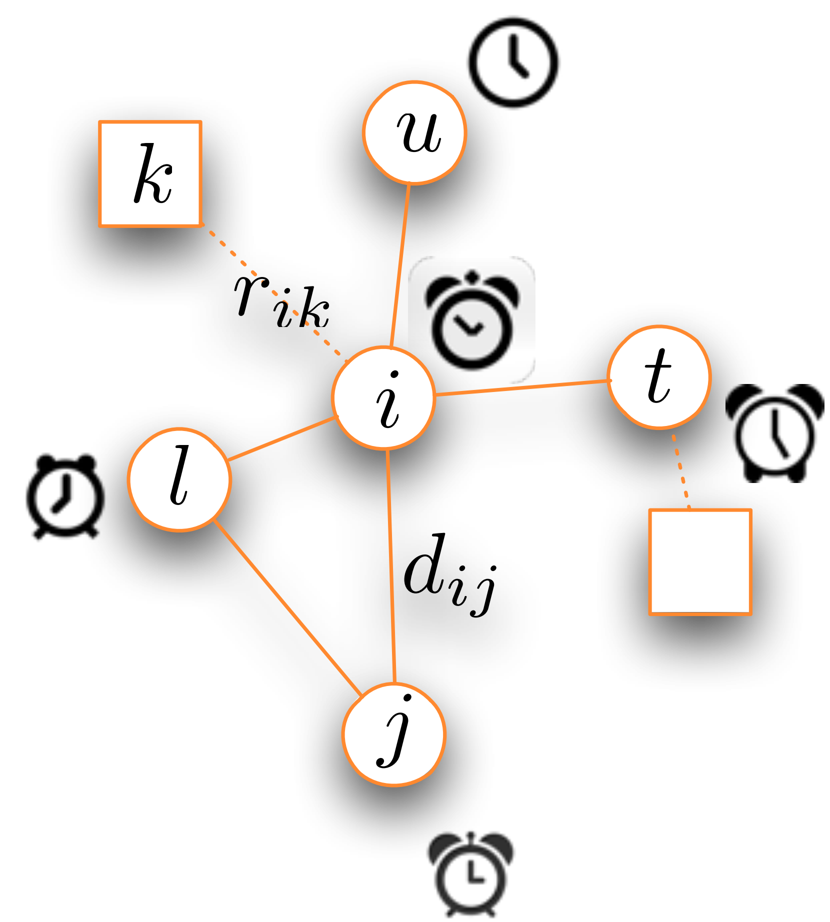
$$T_i \sim PoissonProcess(\lambda)$$

$$\{Z_i^{k+1} - Z_i^k\} \sim Exp(\lambda)$$

$$\Updownarrow$$

$$T \sim PoissonProcess(N\lambda)$$

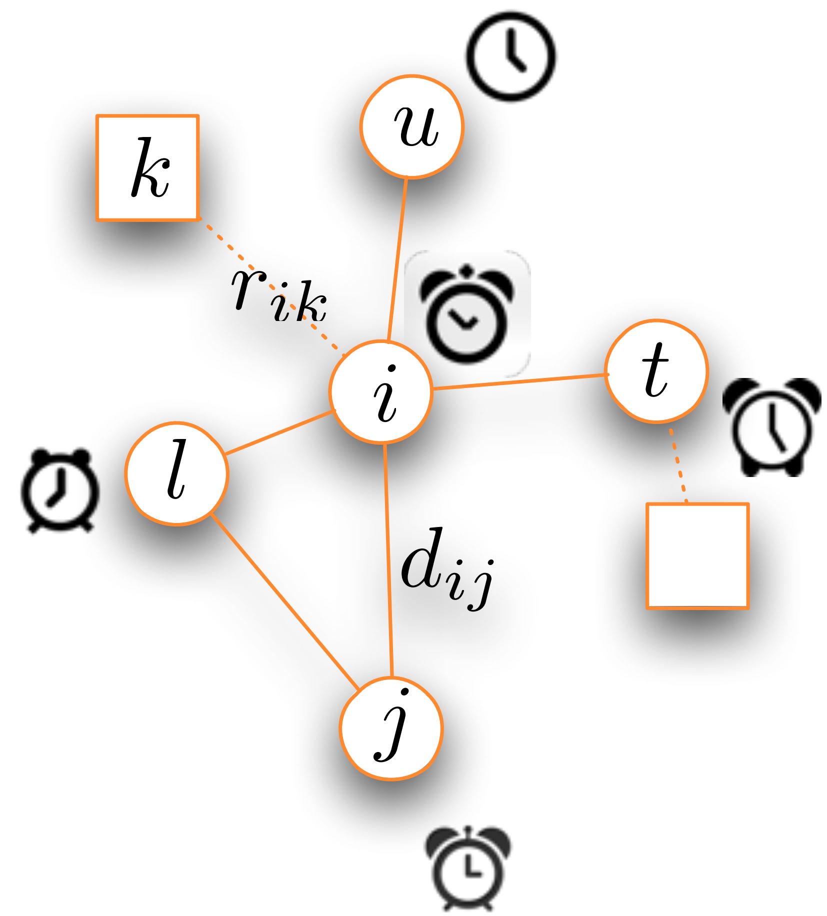
$$\{Z^{k+1} - Z^k\} \sim Exp(N\lambda)$$



Asynchronous time model

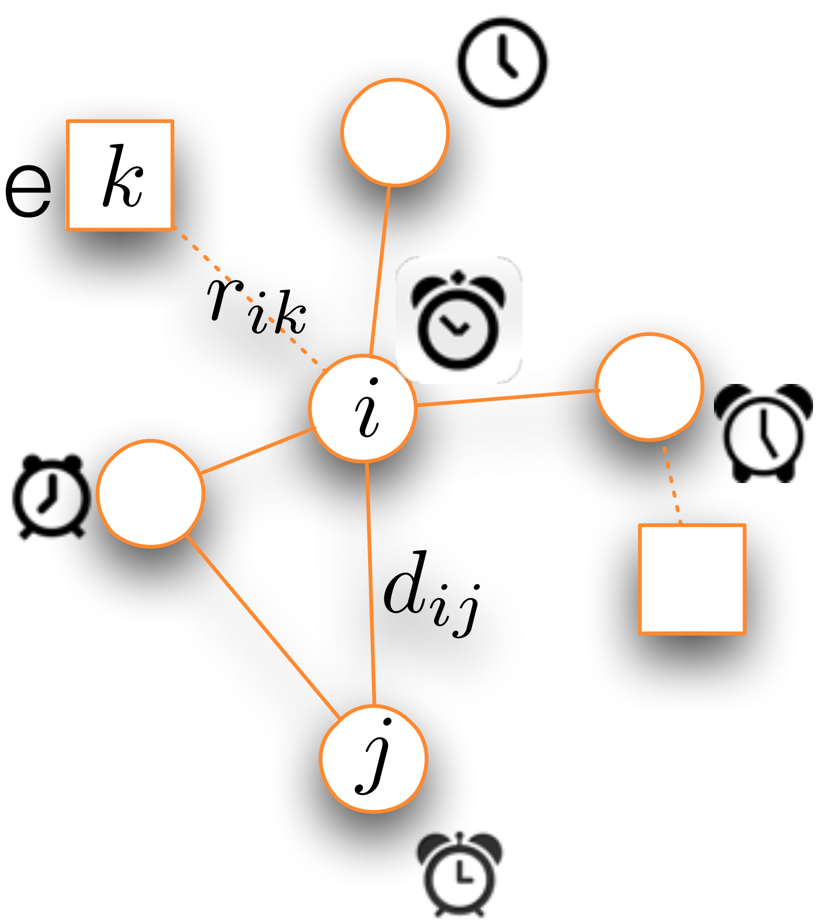
$\xi_k \in \mathcal{V}$

$\xi_k \sim Uniform(\{1, \cdots, N\})$



Asynchronous time algorithms

Algorithm I: The cost is minimized along one of the coordinates at a time



Algorithm II: One gradient step

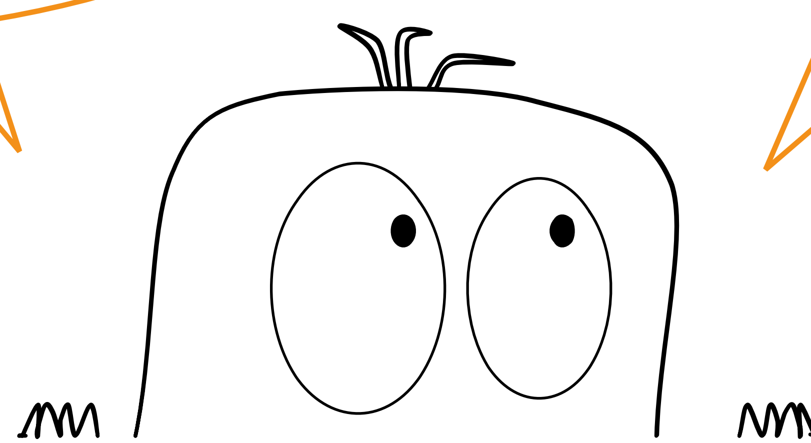
Asynchronous time algorithms

Almost
sure convergence
(alg.I and II)

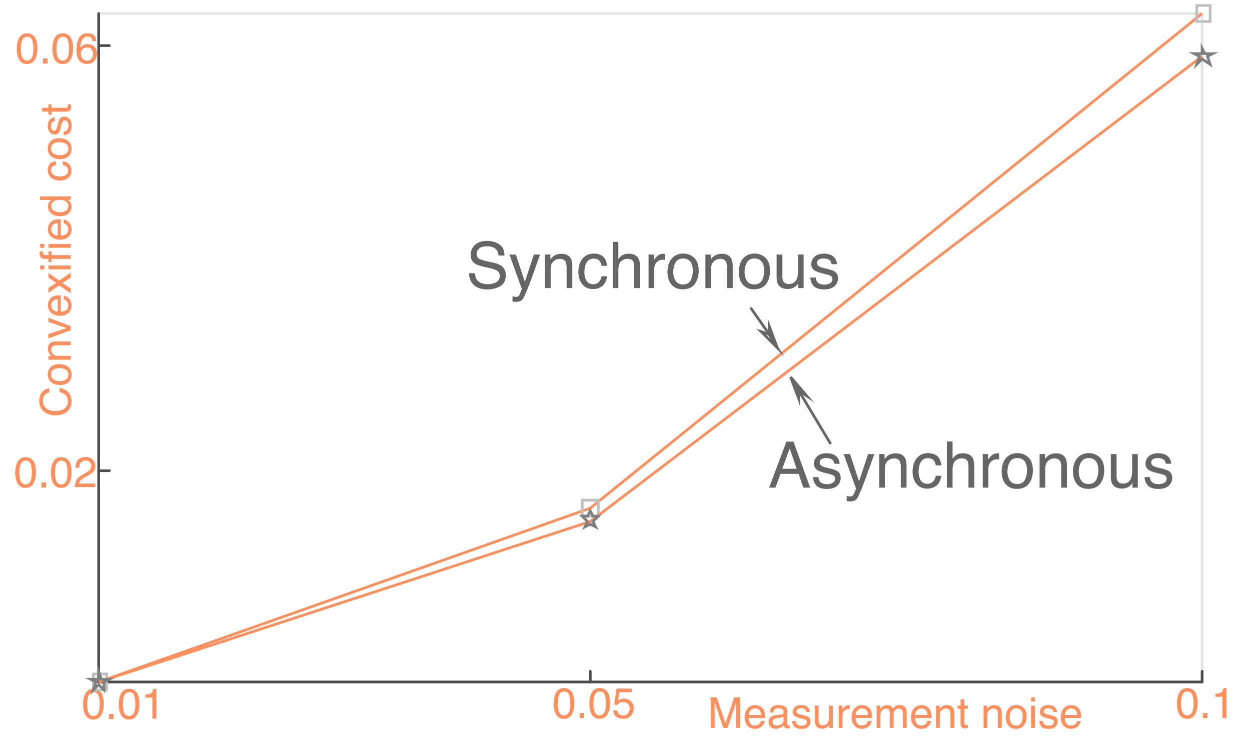
$$\lim_{k\to\infty} \mathrm{d}_{\mathcal{X}^\star}\left(x(k)\right) = 0$$

Almost
sure convergence to a
point (alg. II)

$$x(k) \to x^\star$$



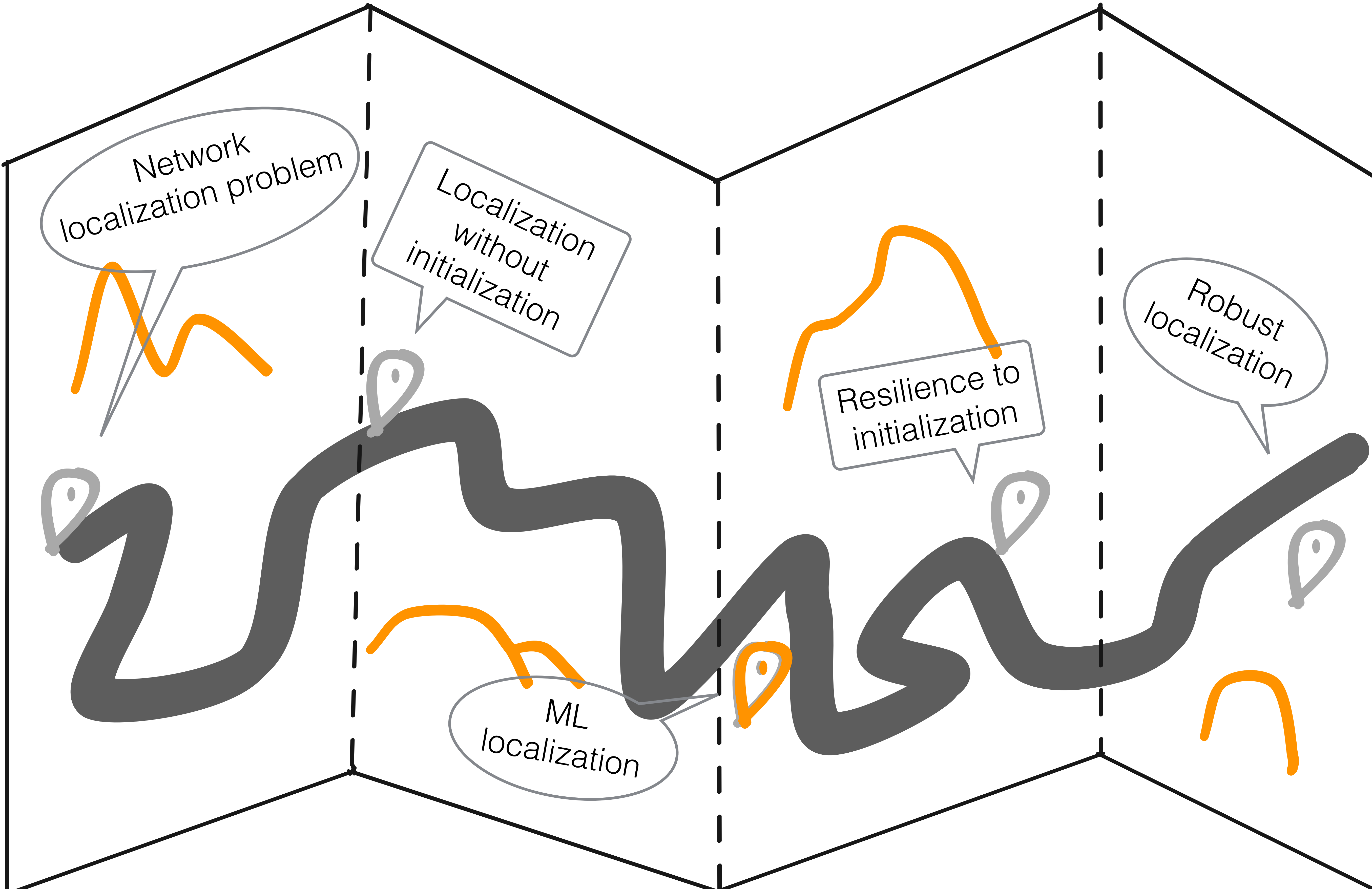
Experimental results



Main contributions

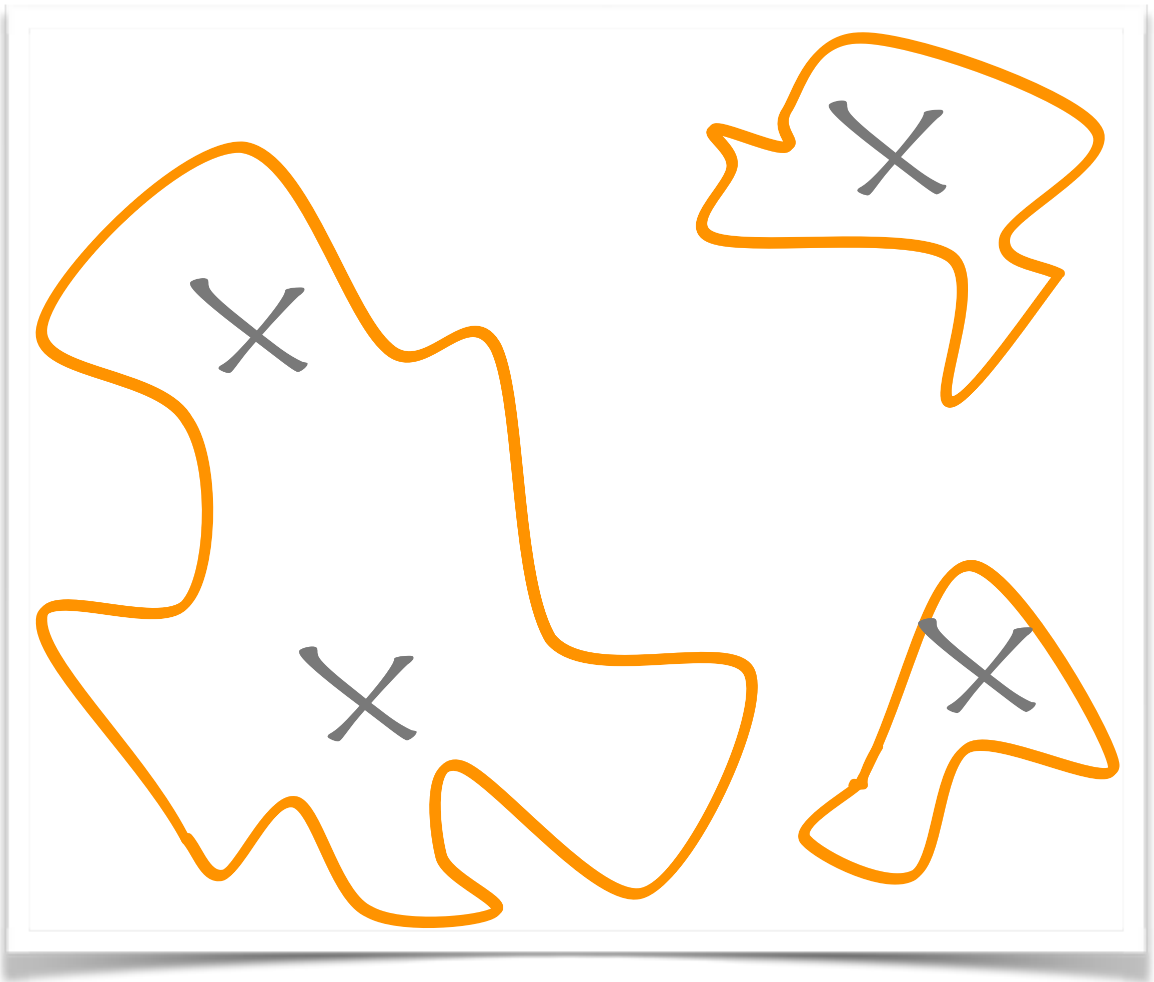
- Optimal gradient distributed algorithm in a synchronous time model to minimize the convexified function;
- Asynchronous randomized algorithm, with proof of almost sure convergence, and expected number of iterations to achieve a desired accuracy;
- Extension for range and angle measurements (submitted to the TSP).

IEEE TSP, 2015


Network localization problem
Localization without initialization
Resilience to initialization
Robust localization
ML localization

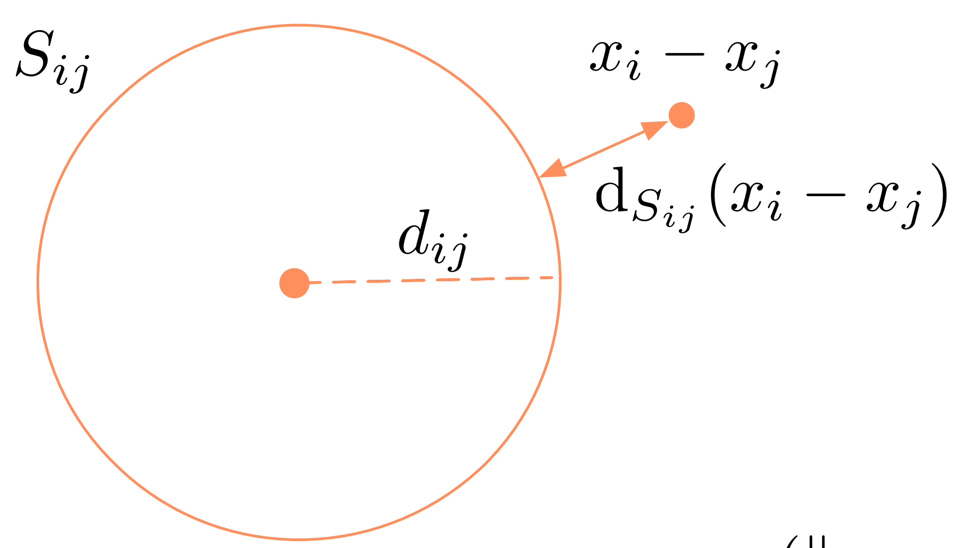
ML localization

Imagine you have a rough idea of where your nodes are located, but you need more precision…



The problem

$$\underset{x}{\text{minimize}} \sum_{i \sim j} \frac{1}{2} \left(\|x_i - x_j\| - d_{ij} \right)^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \left(\|x_i - a_k\| - r_{ik} \right)^2$$



$$\underbrace{\left(\|x_i - x_j\| - d_{ij} \right)^2}_{\mathrm{d}^2_{S_{ij}}(x_i - x_j)} = \underset{y}{\text{minimize}} \; \|x_i - x_j - y\|^2$$

$$\text{subject to } \|y\| = d_{ij}$$

Nonconvexities on the constraints

$$
\begin{aligned}
\underset{x_i, y_{ij}, w_{ik}}{\text{minimize}} \; & \sum_{i \sim j} \frac{1}{2} \|x_i - x_j - y_{ij}\|^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \|x_i - a_k - w_{ik}\|^2 \\
\text{subject to} \; & \|y_{ij}\| = d_{ij}, \|w_{ik}\| = r_{ik}
\end{aligned}
$$

Recognize a quadratic cost

$$\underset{x_i, y_{ij}, w_{ik}}{\text{minimize}} \sum_{i \sim j} \frac{1}{2} \|x_i - x_j - y_{ij}\|^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \|x_i - a_k - w_{ik}\|^2$$
$$\text{subject to } \|y_{ij}\| = d_{ij}, \|w_{ik}\| = r_{ik}$$

$$\underset{z}{\text{minimize}}\ f(z) = \frac{1}{2} z^T M z - b^T z$$
$$\text{subject to } z \in \mathcal{Z}$$

Recognize a quadratic cost

$$\begin{aligned}&\underset{x_i, y_{ij}, w_{ik}}{\text{minimize}} \ \sum_{i\sim j} \frac{1}{2}\|x_i - x_j - y_{ij}\|^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2}\|x_i - a_k - w_{ik}\|^2 \\ &\text{subject to } \|y_{ij}\| = d_{ij}, \|w_{ik}\| = r_{ik}\end{aligned}$$

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$$\mathcal{Z} = \{z = (x, y, w) : \|y_{ij}\| = d_{ij}, i \sim j, \|w_{ik}\| = r_{ik}, i \in \mathcal{V}, k \in \mathcal{A}_i\}$$

Recognize a quadratic cost

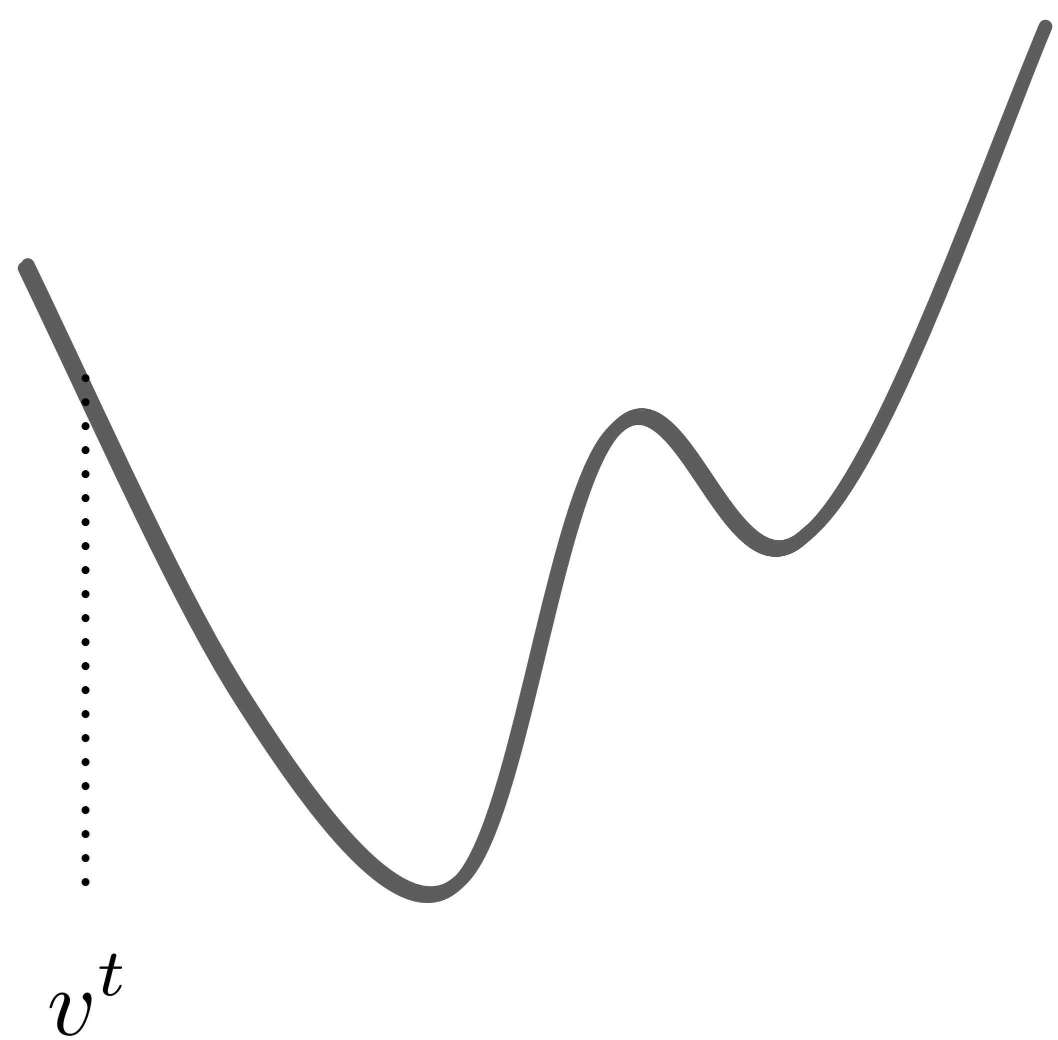
$$\underset{x_i, y_{ij}, w_{ik}}{\text{minimize}} \sum_{i \sim j} \frac{1}{2} \|x_i - x_j - y_{ij}\|^2 + \sum_i \sum_{k \in \mathcal{A}_i} \frac{1}{2} \|x_i - a_k - w_{ik}\|^2$$
$$\text{subject to } \|y_{ij}\| = d_{ij}, \|w_{ik}\| = r_{ik}$$

Not diagonal!

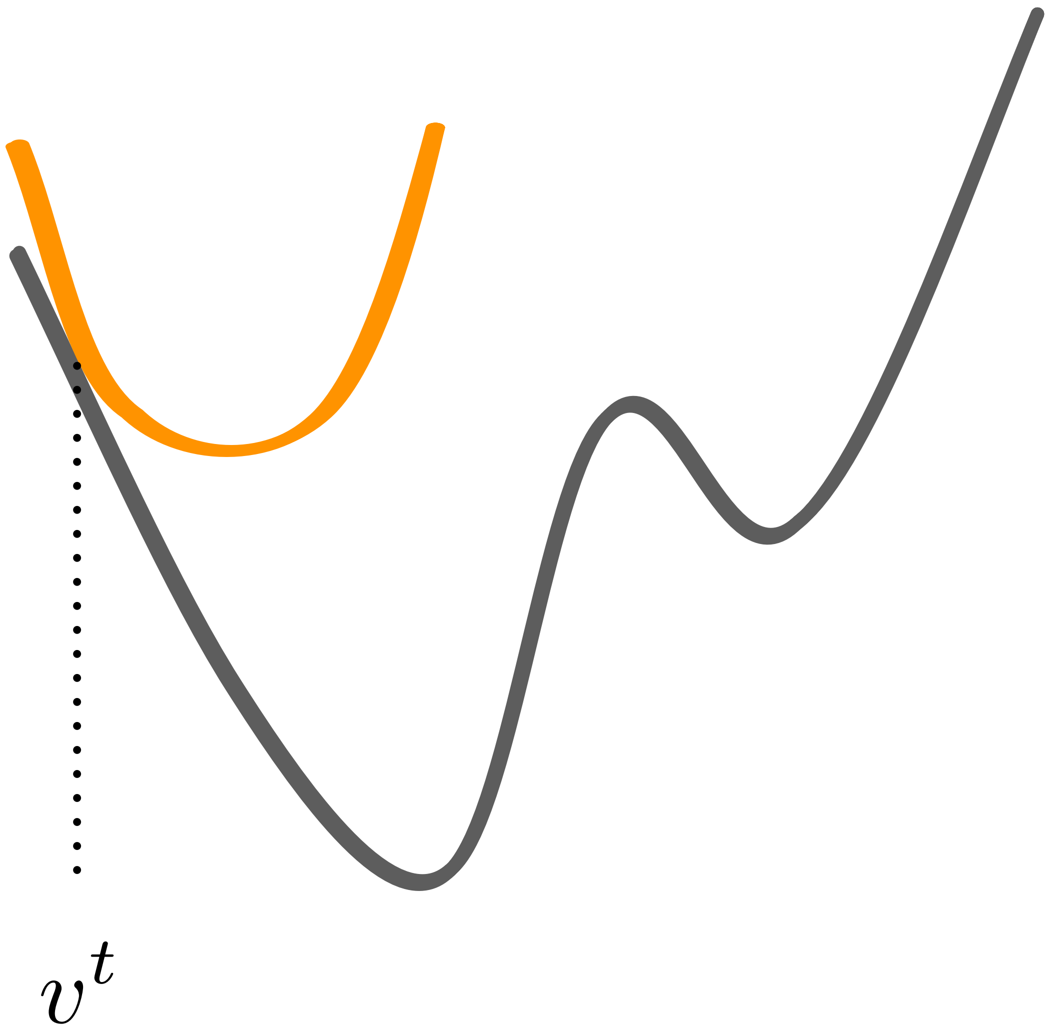
$$\underset{z}{\text{minimize}} \; f(z) = \frac{1}{2} z^T M z - b^T z$$
$$\text{subject to } z \in \mathcal{Z}$$

$$\mathcal{Z} = \{z = (x, y, w) : \|y_{ij}\| = d_{ij}, i \sim j, \|w_{ik}\| = r_{ik}, i \in \mathcal{V}, k \in \mathcal{A}_i\}$$

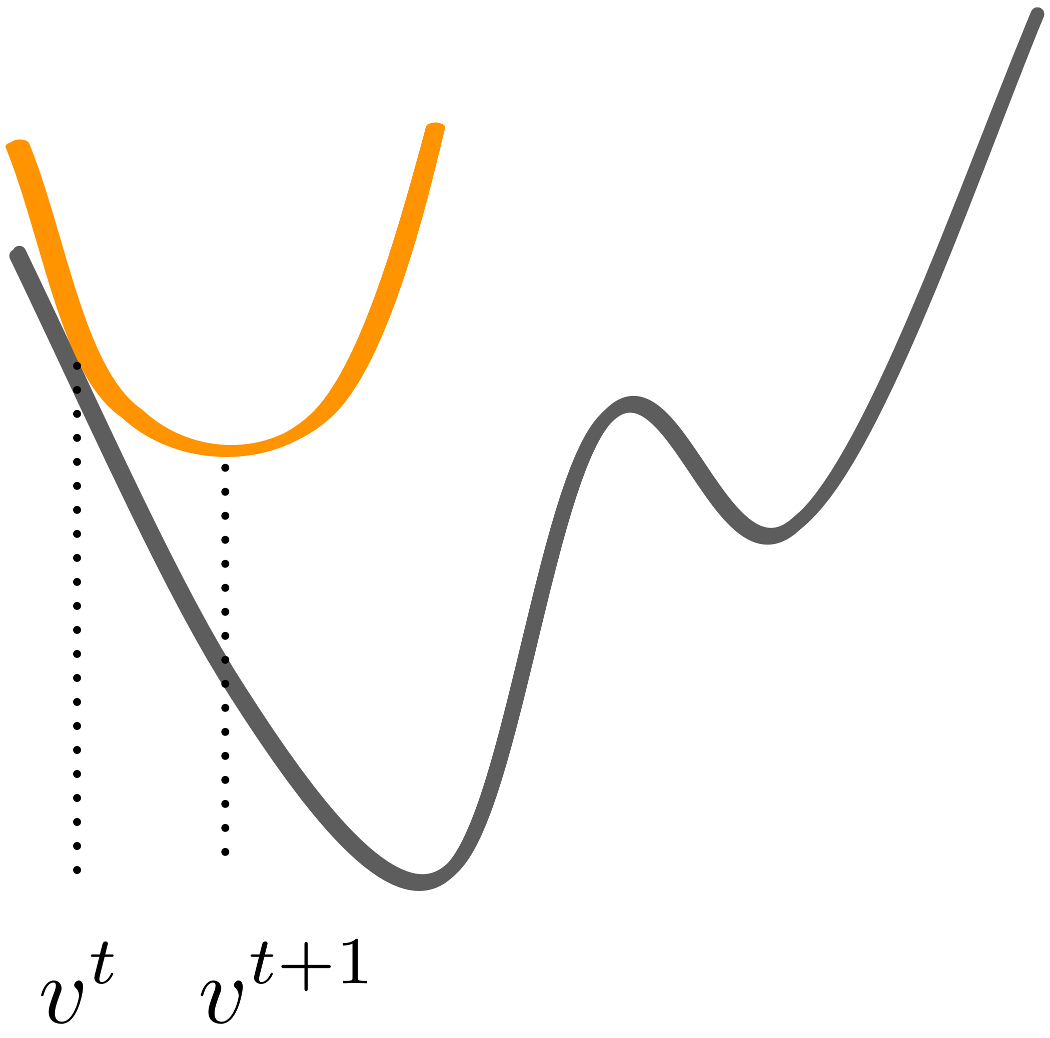
Majorization-Minimization



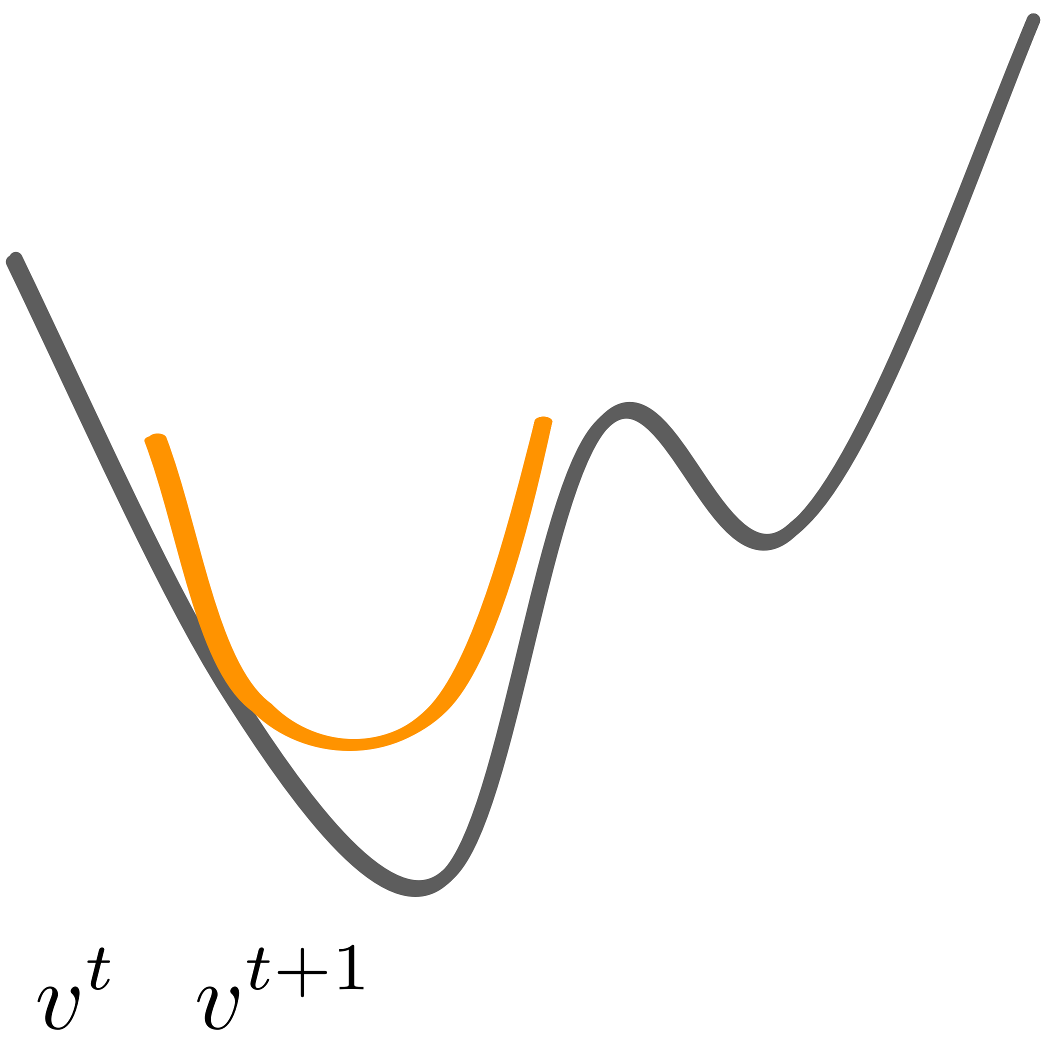
Majorization-Minimization



Majorization-Minimization



Majorization-Minimization



Majorization-Minimization

Majorization-Minimization

$$\underset{z}{\text{minimize}}\ f(z) = \frac{1}{2} z^T M z - b^T z$$
$$\text{subject to } z \in \mathcal{Z}$$

Majorization-Minimization

$$\underset{z}{\text{minimize}}\ f(z) = \frac{1}{2} z^T M z - b^T z$$
$$\text{subject to } z \in \mathcal{Z}$$

$$f(z) \leq f(z^t) + \left\langle \nabla f(z^t), z - z^t \right\rangle + \frac{L}{2} \left\| z - z^t \right\|^2$$

Majorization-Minimization

Diagonal quadratic term

$$f(z) \leq f(z^t) + \langle \nabla f(z^t), z - z^t \rangle + \frac{L}{2} \left\| z - z^t \right\|^2$$

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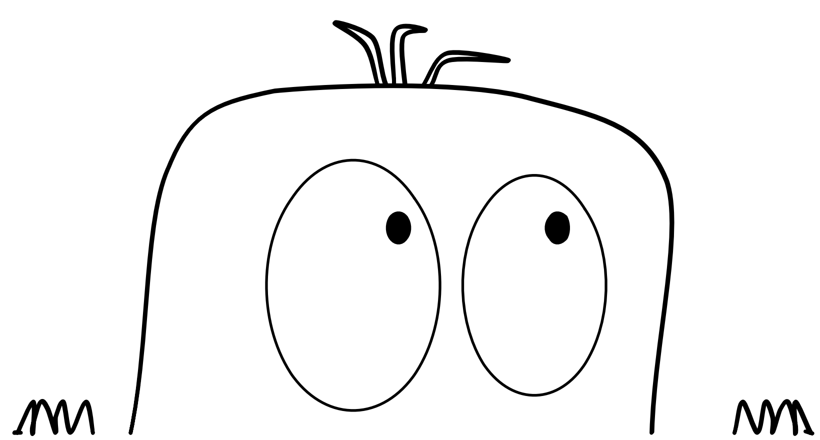
Majorizer decouples the variables and allows for a distributed solution!

Nice properties

Cost guaranteed not to increase at each iteration

No parameters to tune

Distributed computations



Every limit point is a stationary point

Experimental results

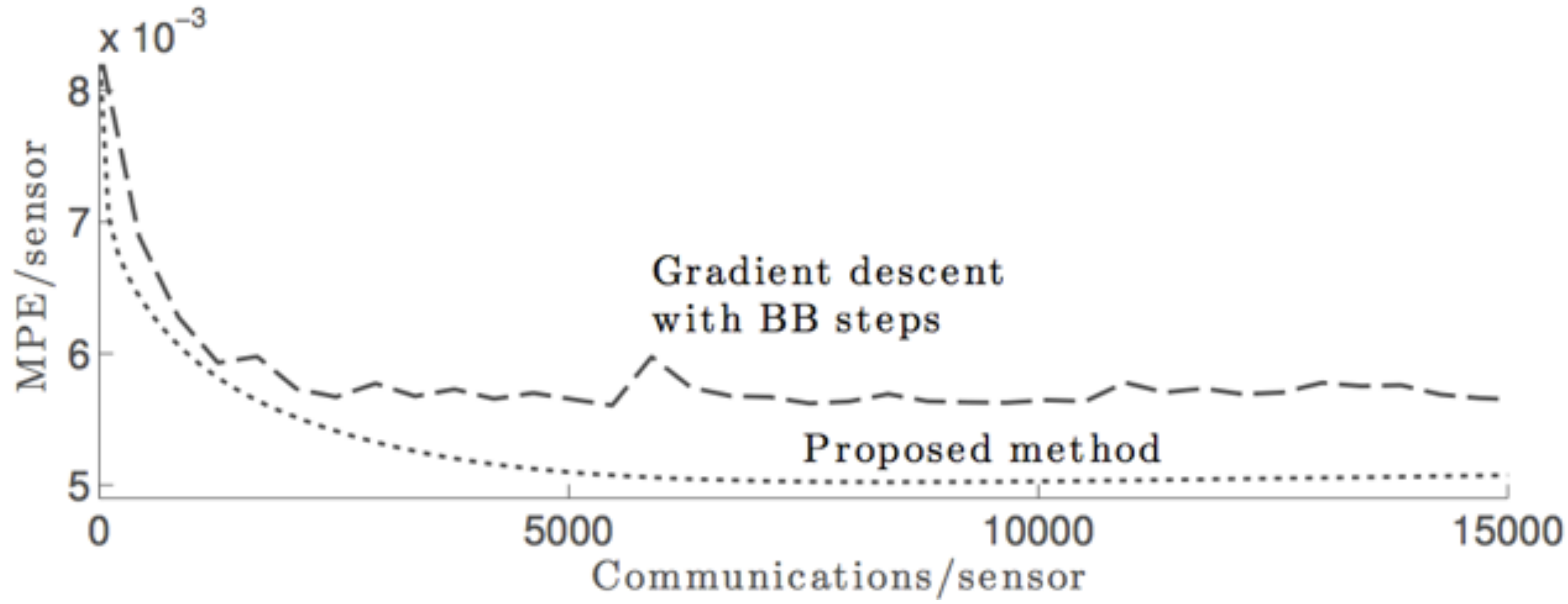
Noise standard deviation	Proposed method	Gradient descent with Barzilai-Borwain steps
0.01	0.0053	0.0059
0.05	0.0143	0.0154
0.10	0.0210	0.0221

$$\text{MPE} = \frac{1}{M} \sum_{m=1}^{M} \|\hat{x}(m) - x^\star\|$$

In a square with 1 Km sides, we improve the accuracy of the benchmark by about 1m per sensor, even under high power noise.

Calafiore, *et al.*, CDC 2010

Experimental results

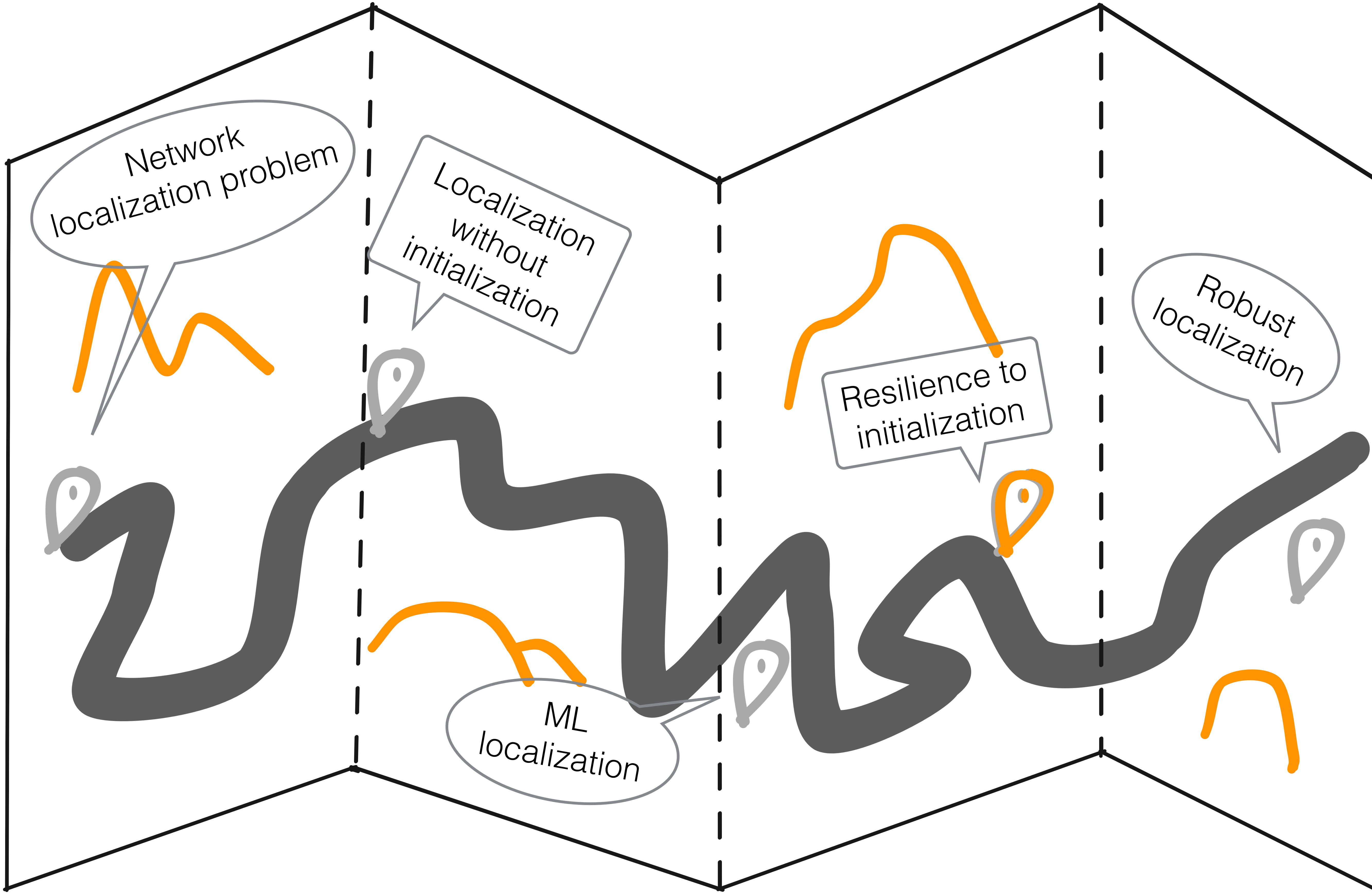


Our proposed method improves the state of the art method by about 60 cm in mean positioning error per sensor, delivering a no surprises, stable progression of the error of the estimates.

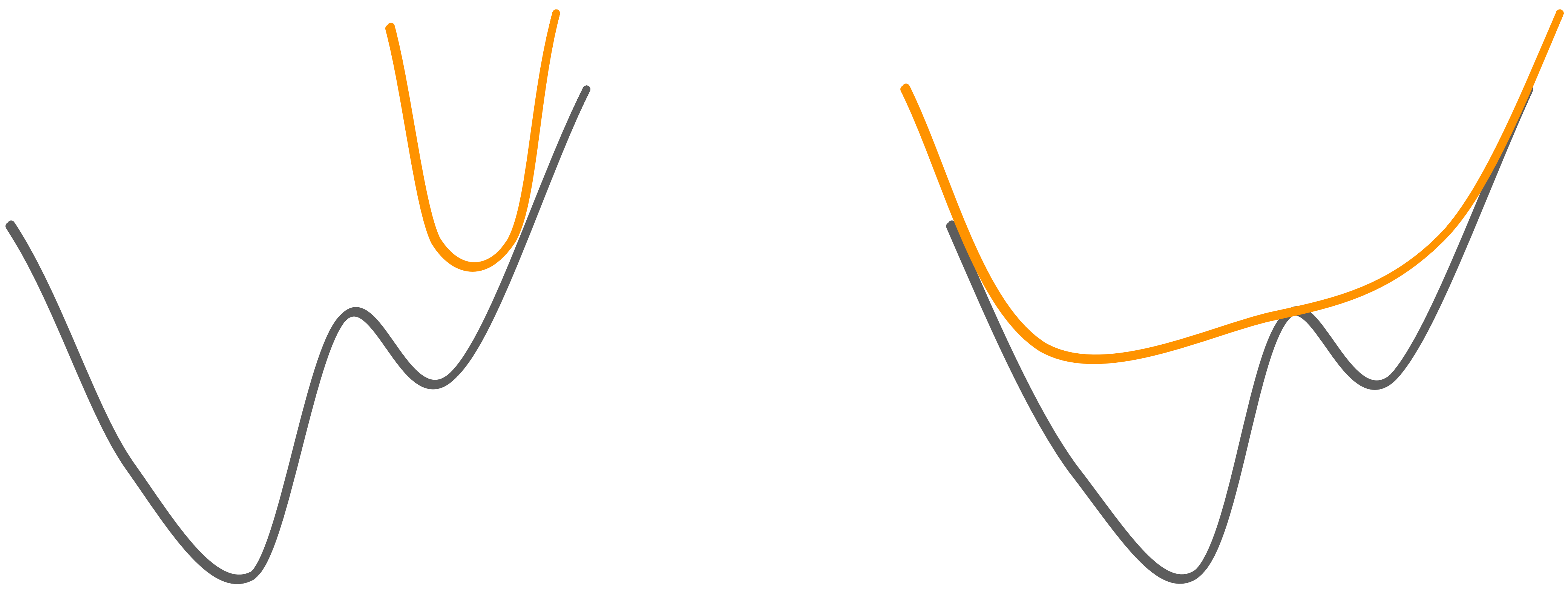
Main contributions

- No parameter tuning;
- Stable algorithm: the cost value decreases at each iteration;
- Simple to implement, distributed, and efficient algorithm for the nonconvex ML estimator.

IEEE GlobalSIP, 2014


Network localization problem
Localization without initialization
ML localization
Resilience to initialization
Robust localization

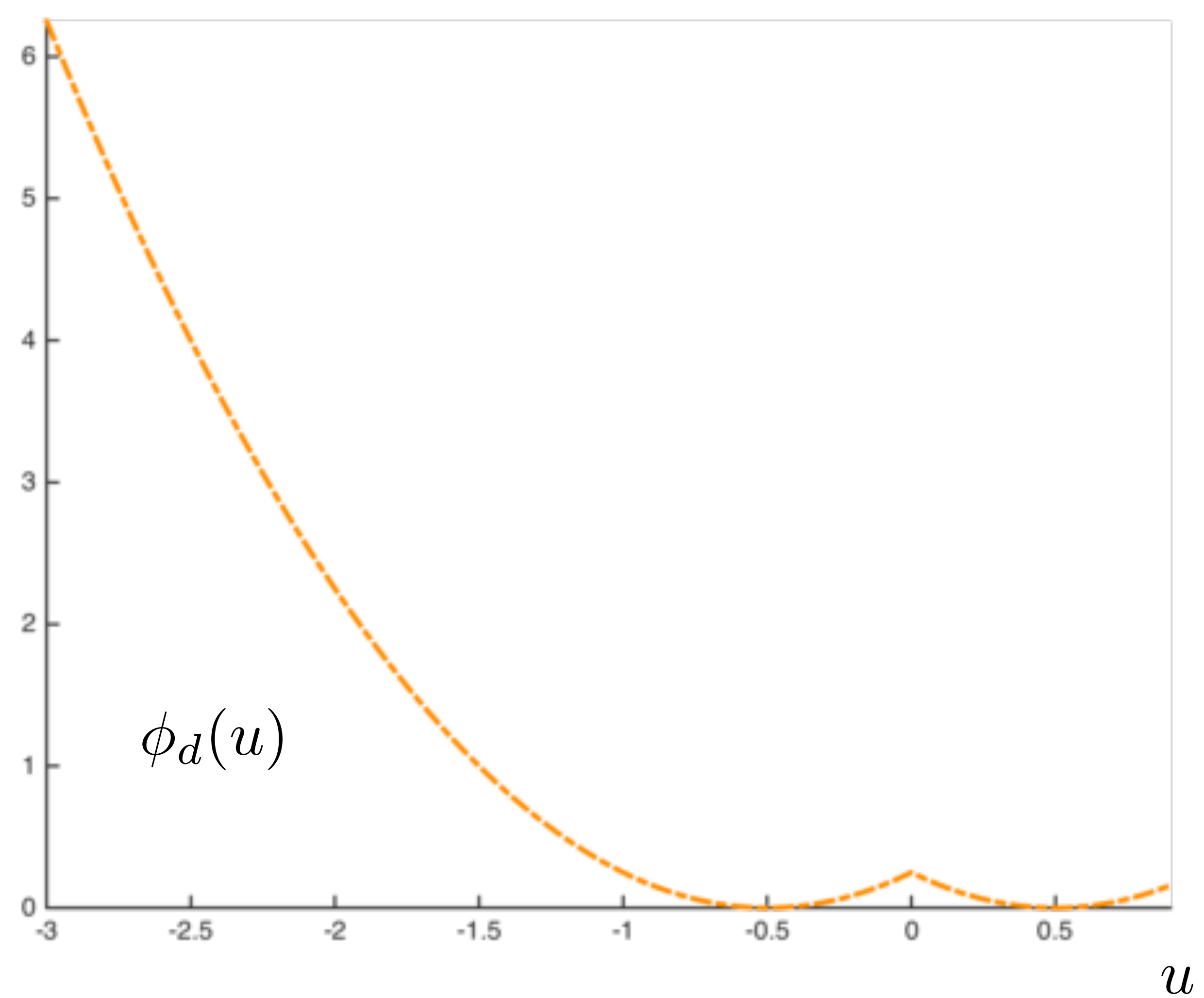
With more computations we can do better



Tighter majorizer

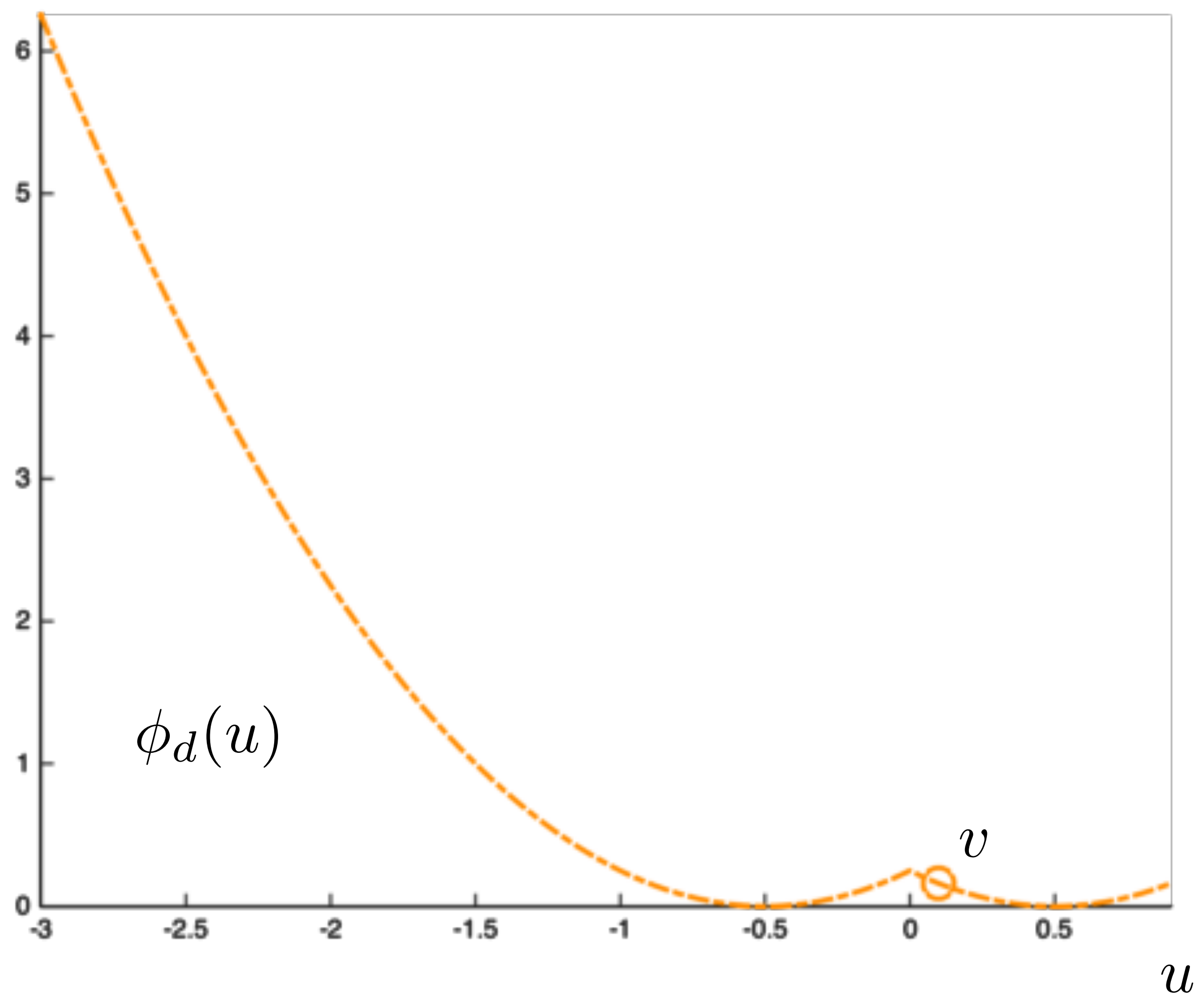
Tighter majorizer

Nonconvex term $\phi_d(u) = (\|u\| - d)^2$



Tighter majorizer

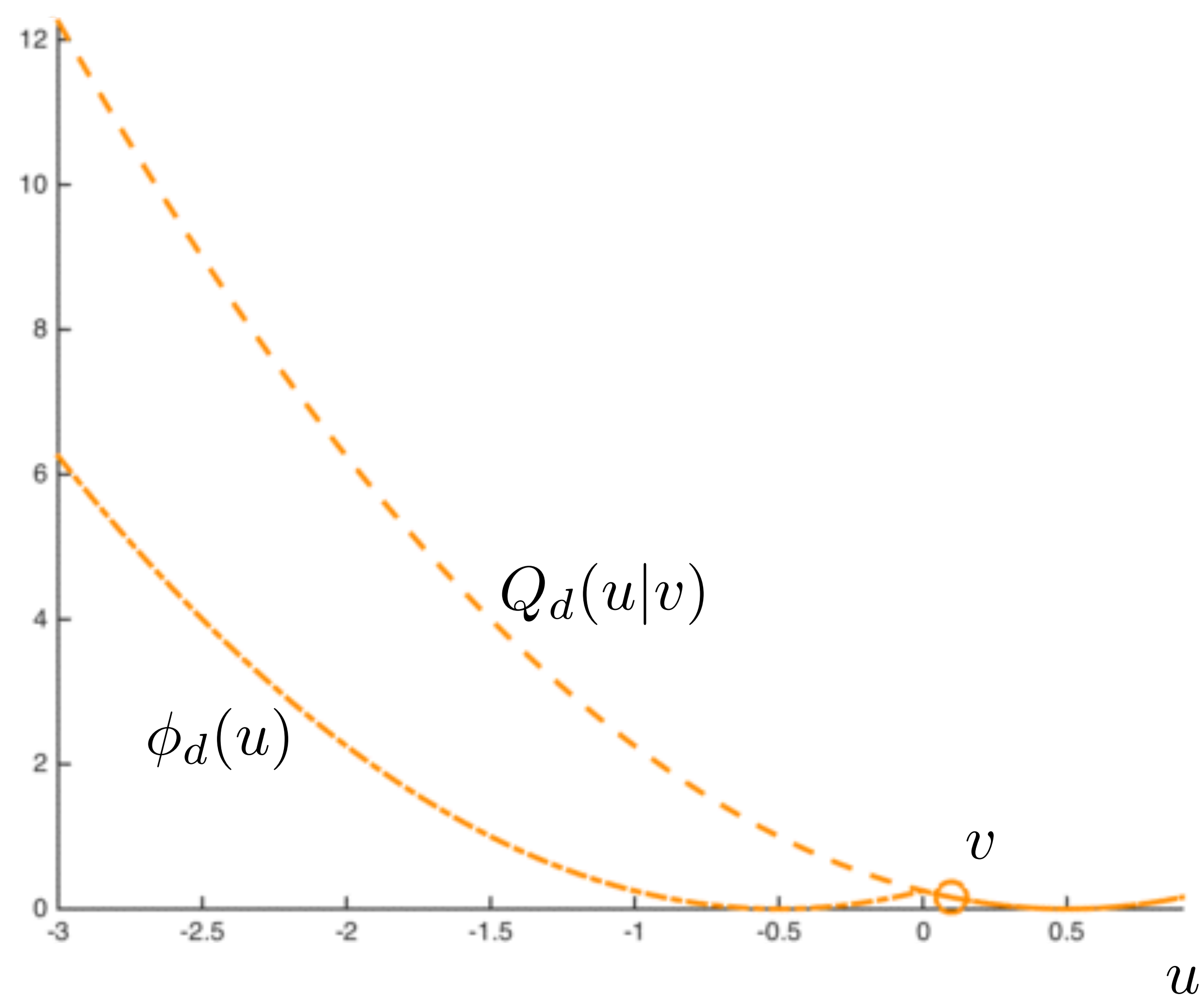
Nonconvex term $\phi_d(u) = (\|u\| - d)^2$



Tighter majorizer

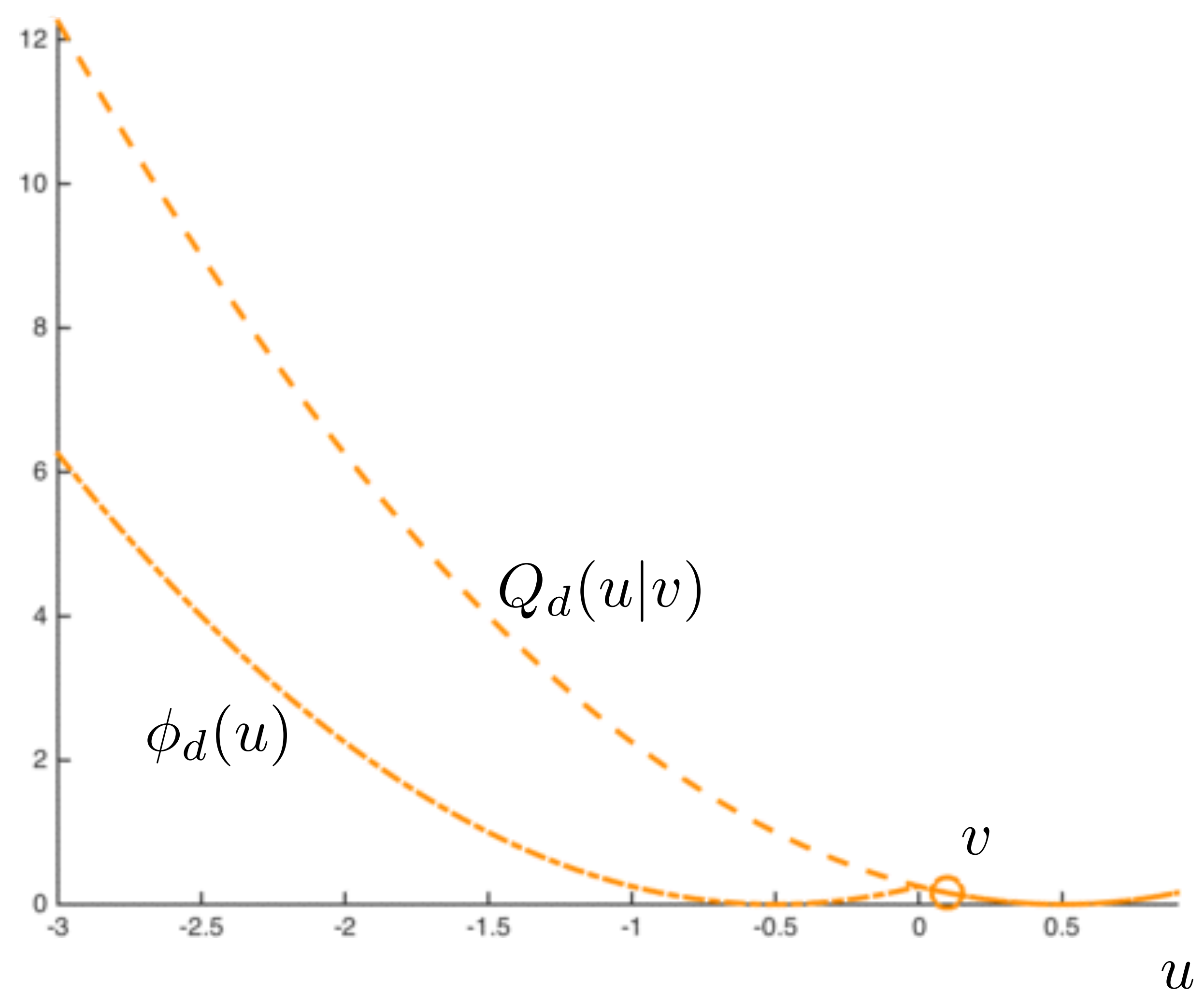
Nonconvex term $\phi_d(u) = (\|u\| - d)^2$

Best quadratic majorizer $Q_d(u|v) = \|u\|^2 + d^2 - 2d\frac{v^\top u}{\|v\|}$



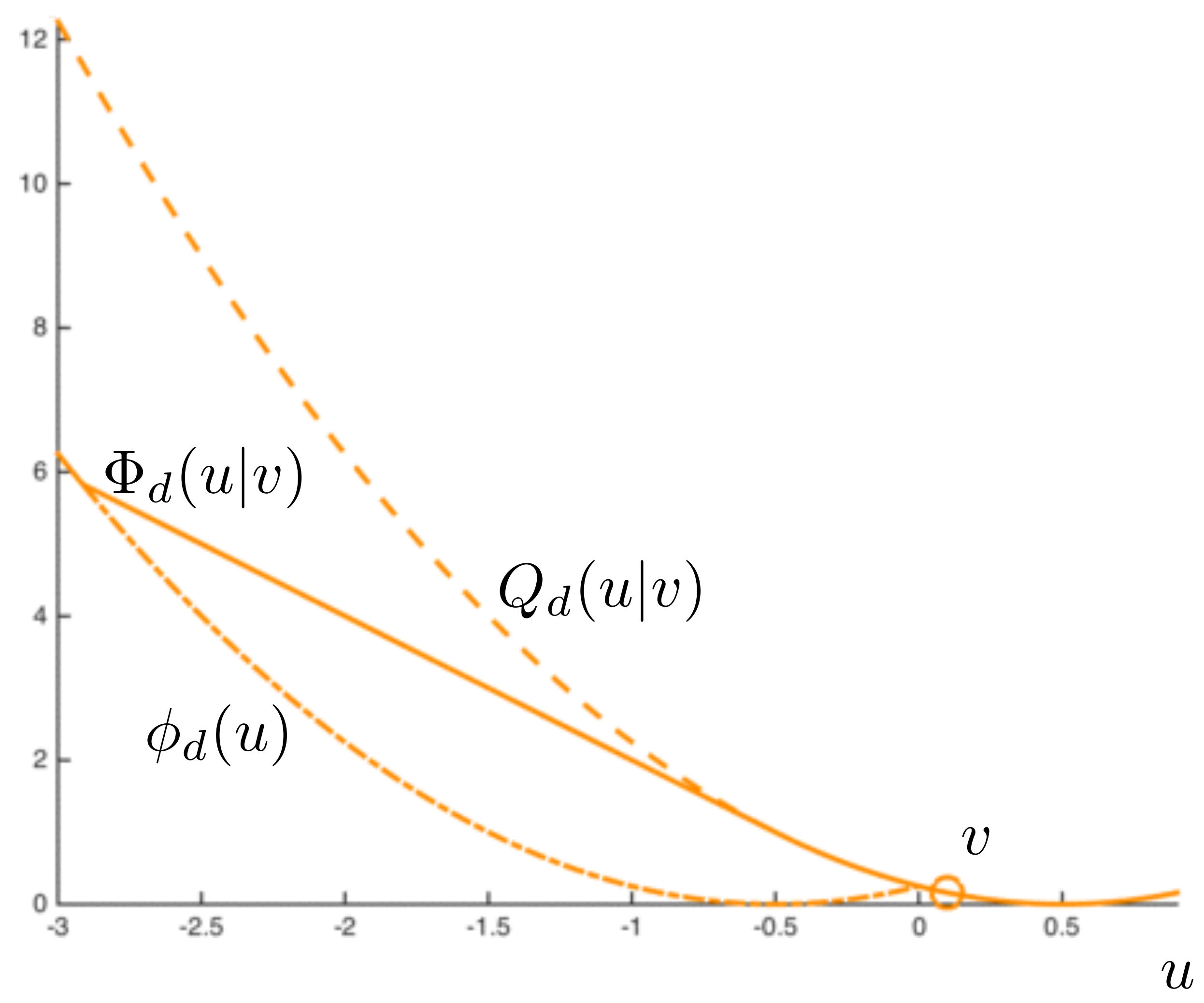
Tighter majorizer

The proposed majorizer



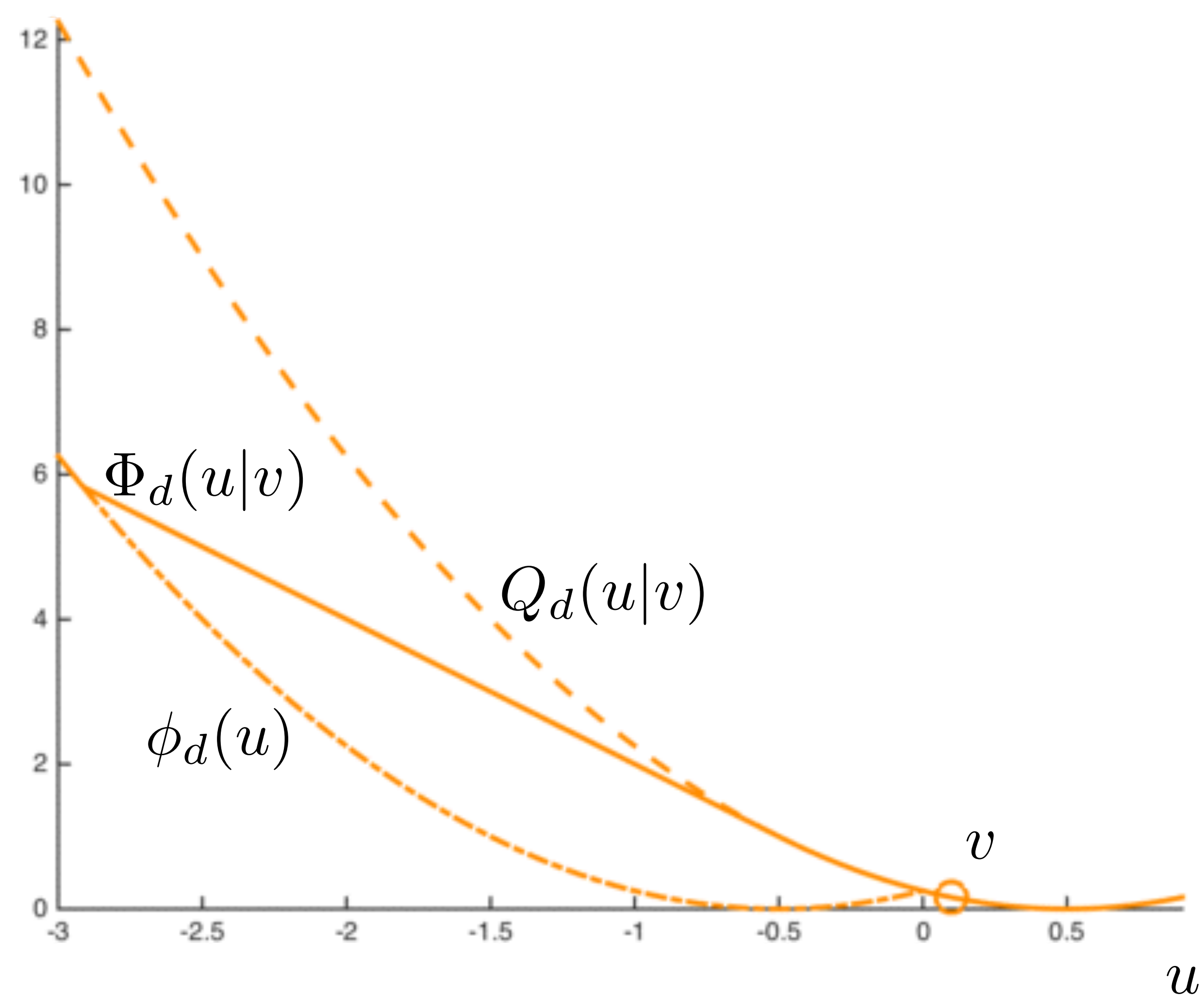
Tighter majorizer

The proposed majorizer



Tighter majorizer

The proposed majorizer $\Phi_d(u|v) = \max\left\{g_d(u), h_d(v^\top u / \|v\| - d)\right\}$



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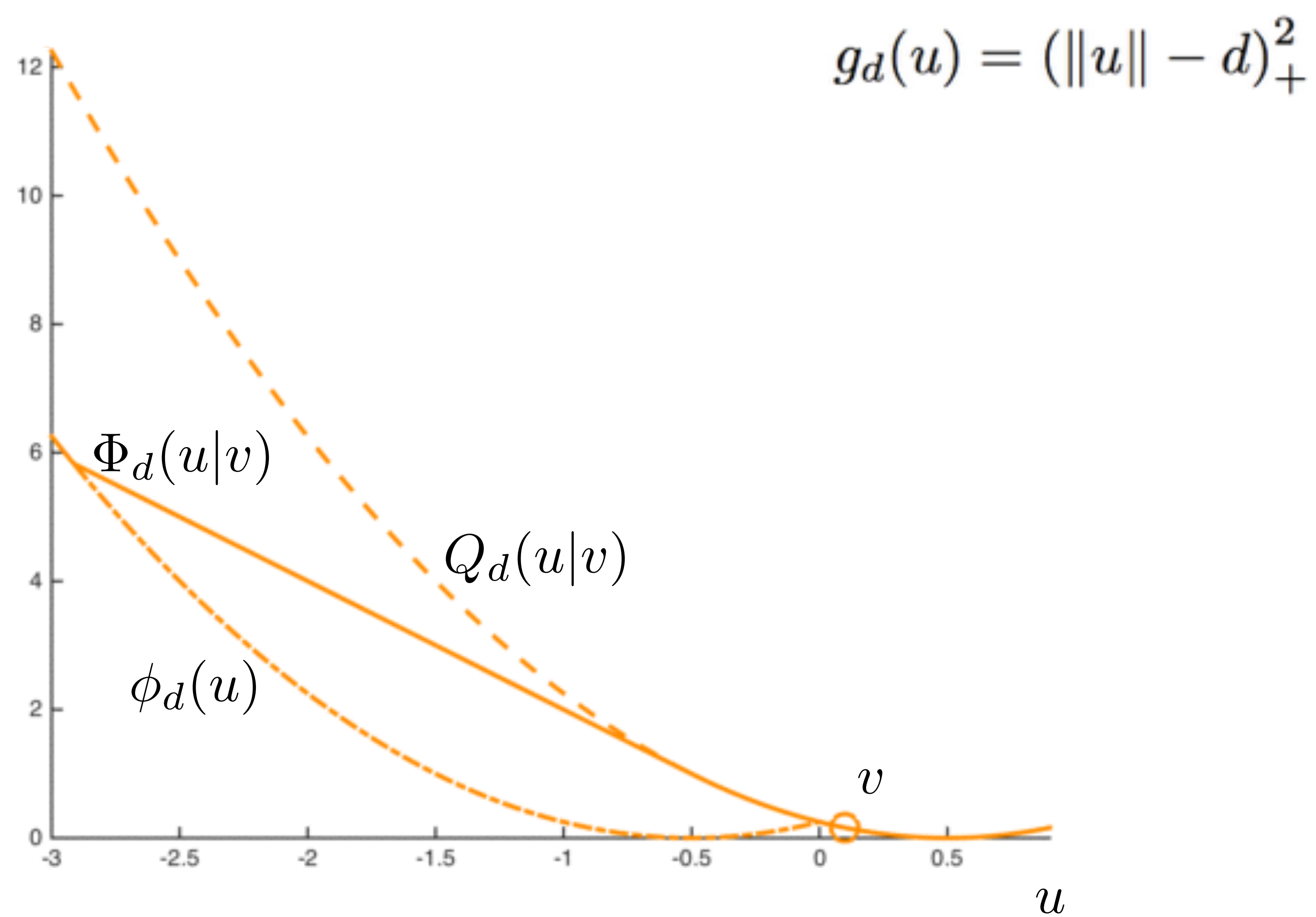
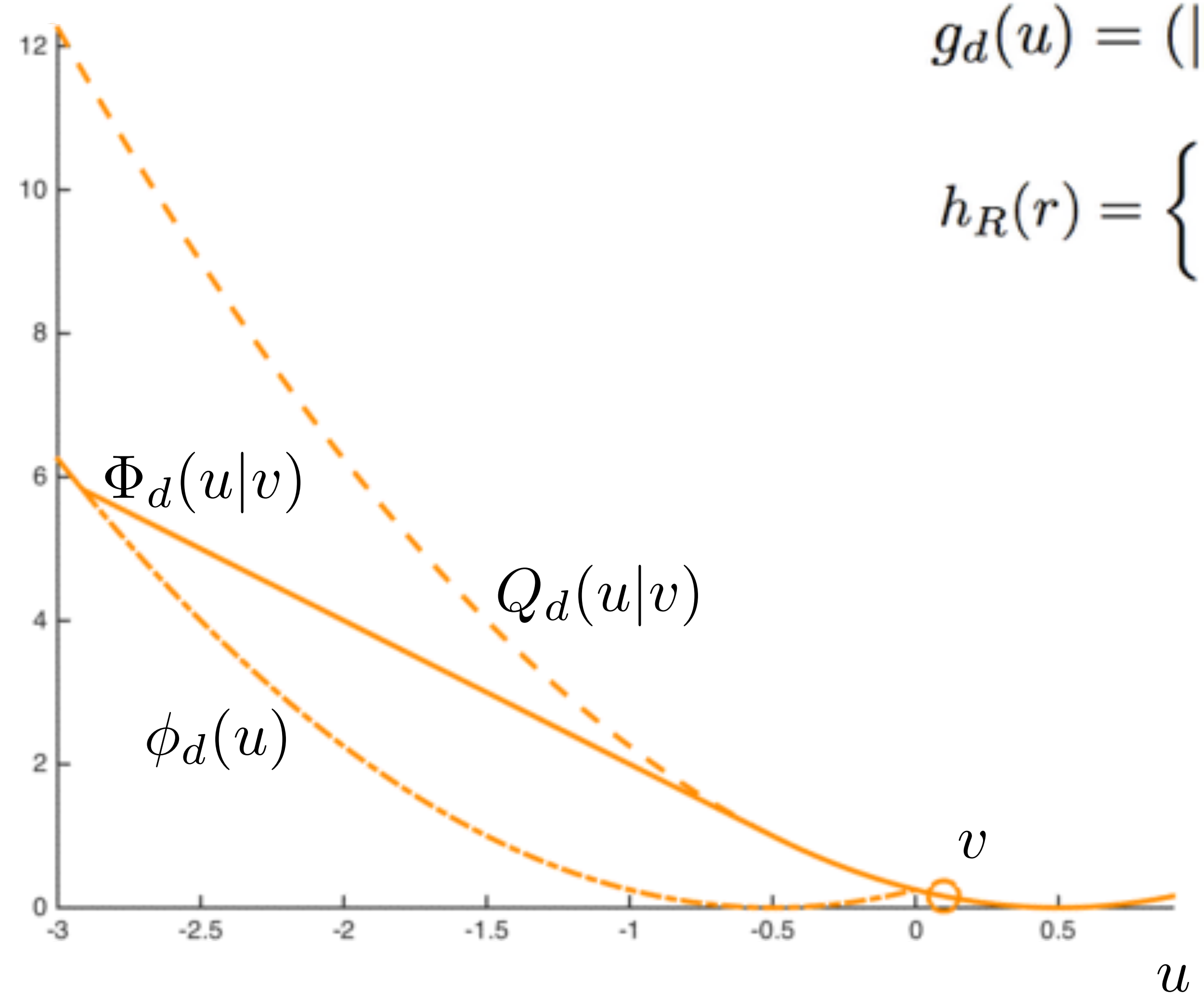


Tighter majorizer

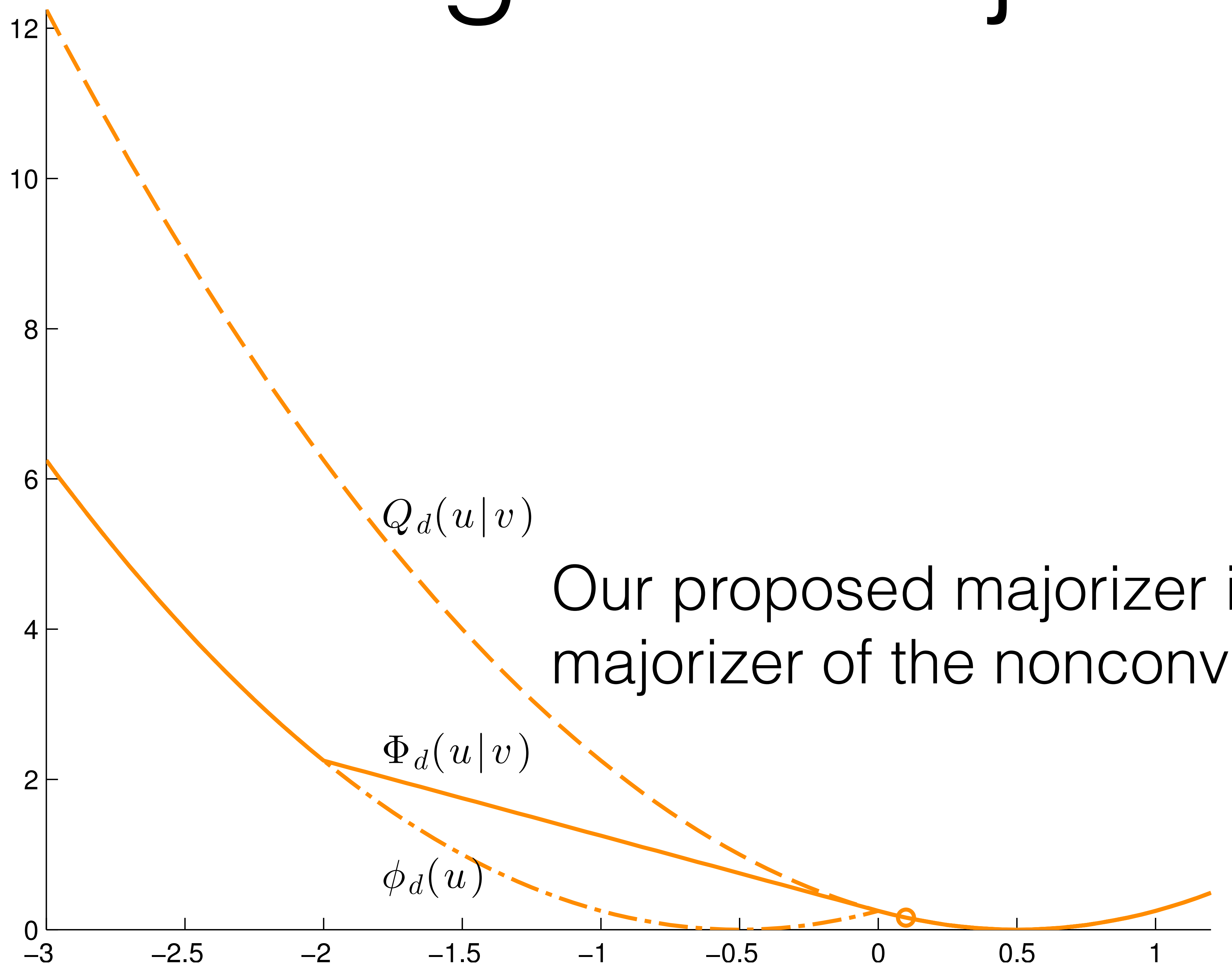
The proposed majorizer $\Phi_d(u|v) = \max\left\{g_d(u), h_d(v^\top u/\|v\| - d)\right\}$

$$g_d(u) = (\|u\| - d)_+^2$$

$$h_R(r) = \begin{cases} 2R|r| - R^2 & \text{if } |r| \geq R \\ r^2 & \text{if } |r| < R \end{cases}$$



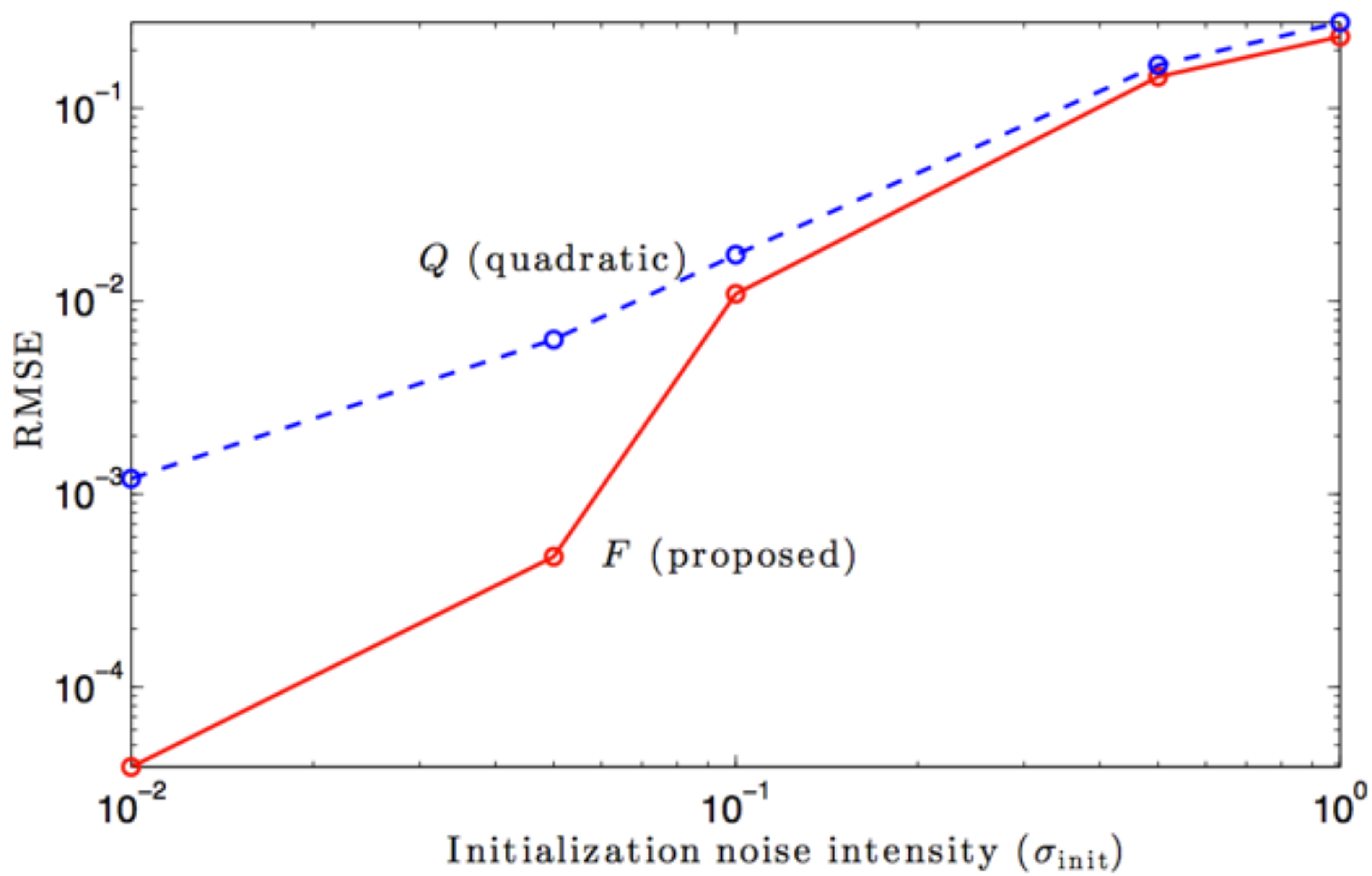
Tighter majorizer



Conjecture

Our proposed majorizer is a tight convex majorizer of the nonconvex function $\phi_d(u)$

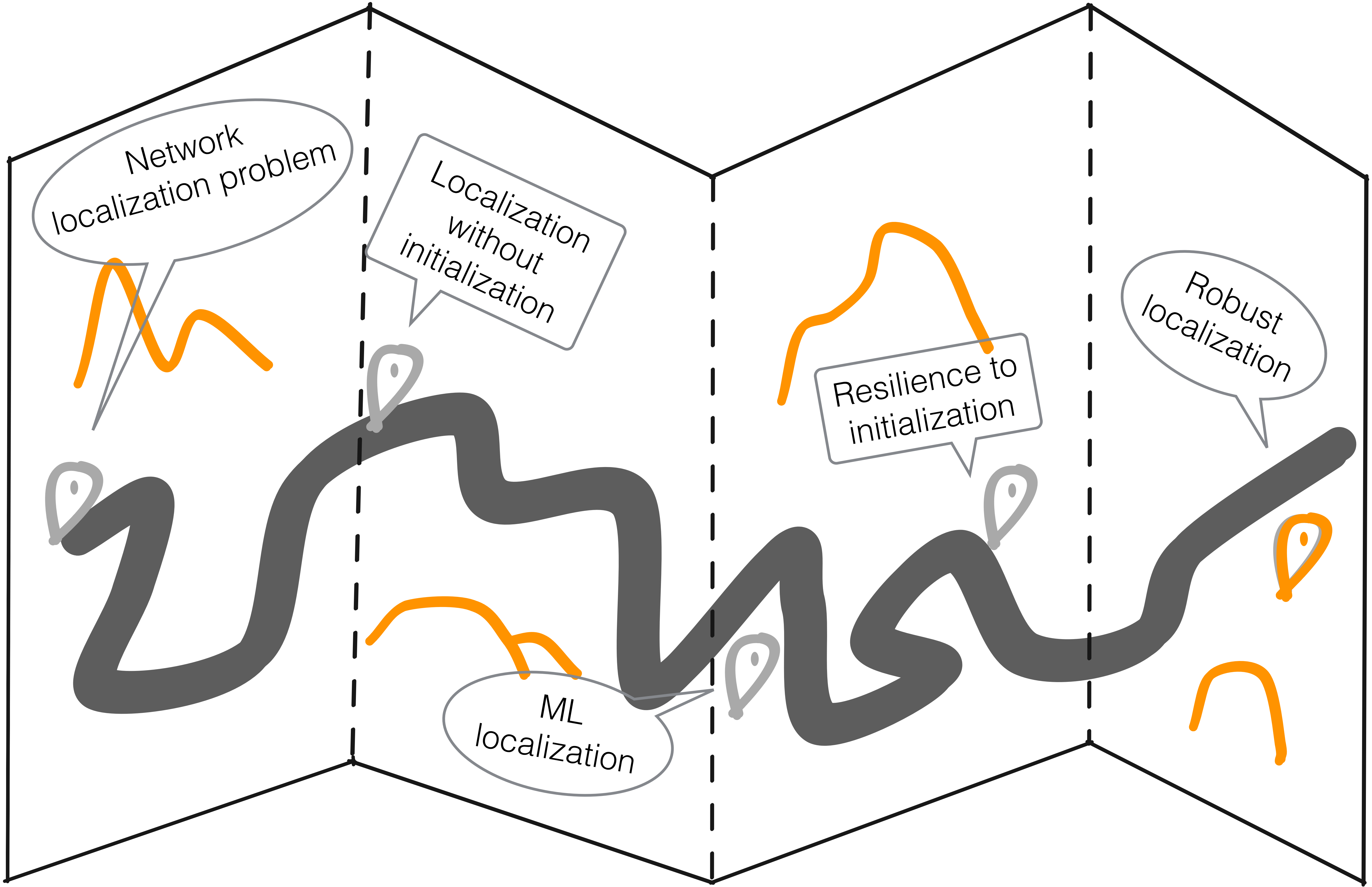
Experimental results on majorization function quality



Main contributions

- A tight convex majorization function for the ML problem;
- Useful for other contexts (e.g. molecular geometry);
- A distributed method to solve each resulting MM problem.

To be submitted


Network localization problem
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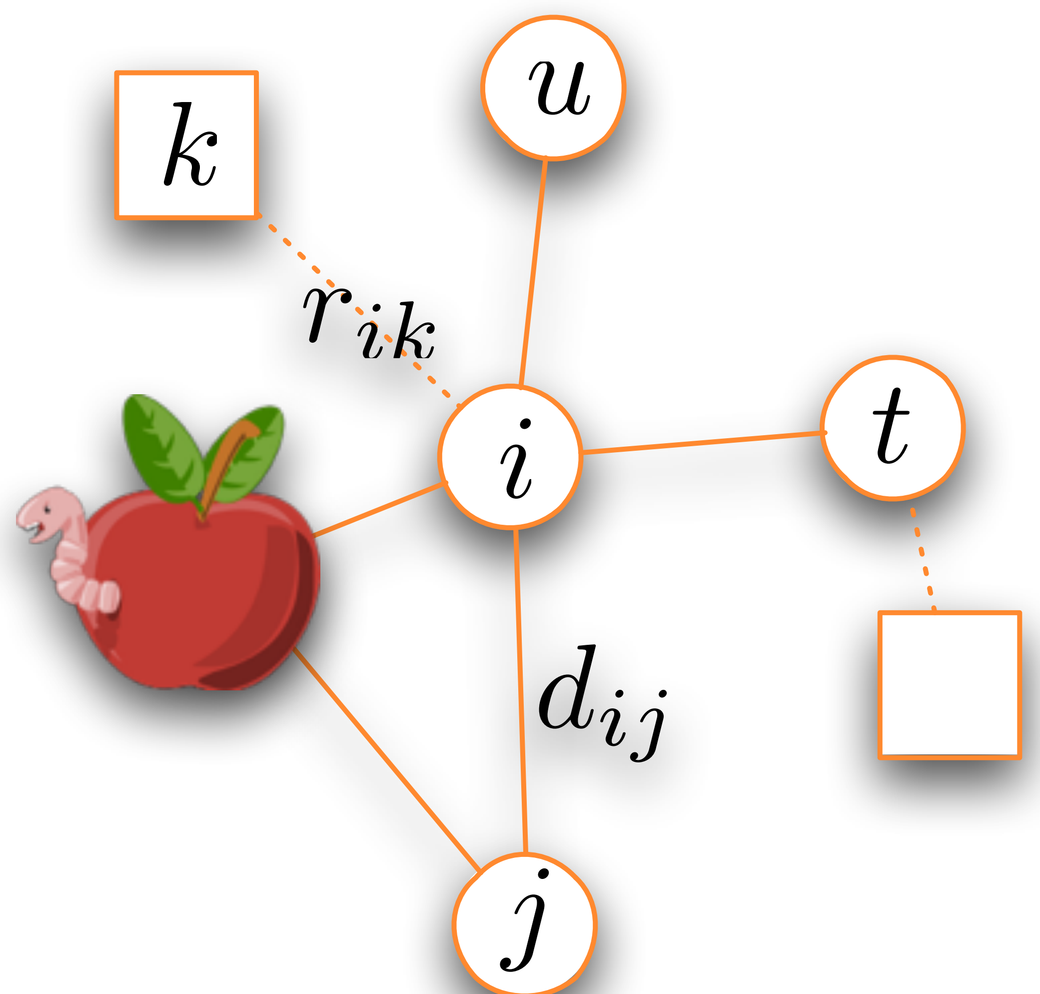
Robust localization

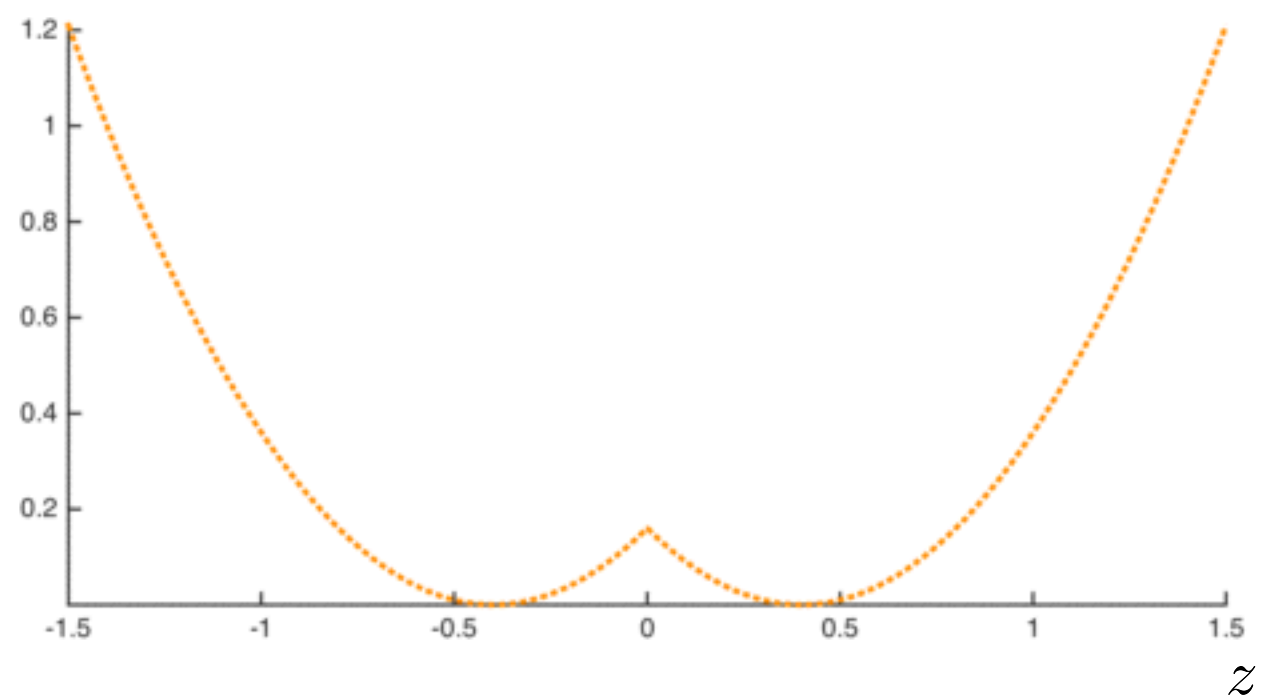
Robust network localization



Dissimilarity measures

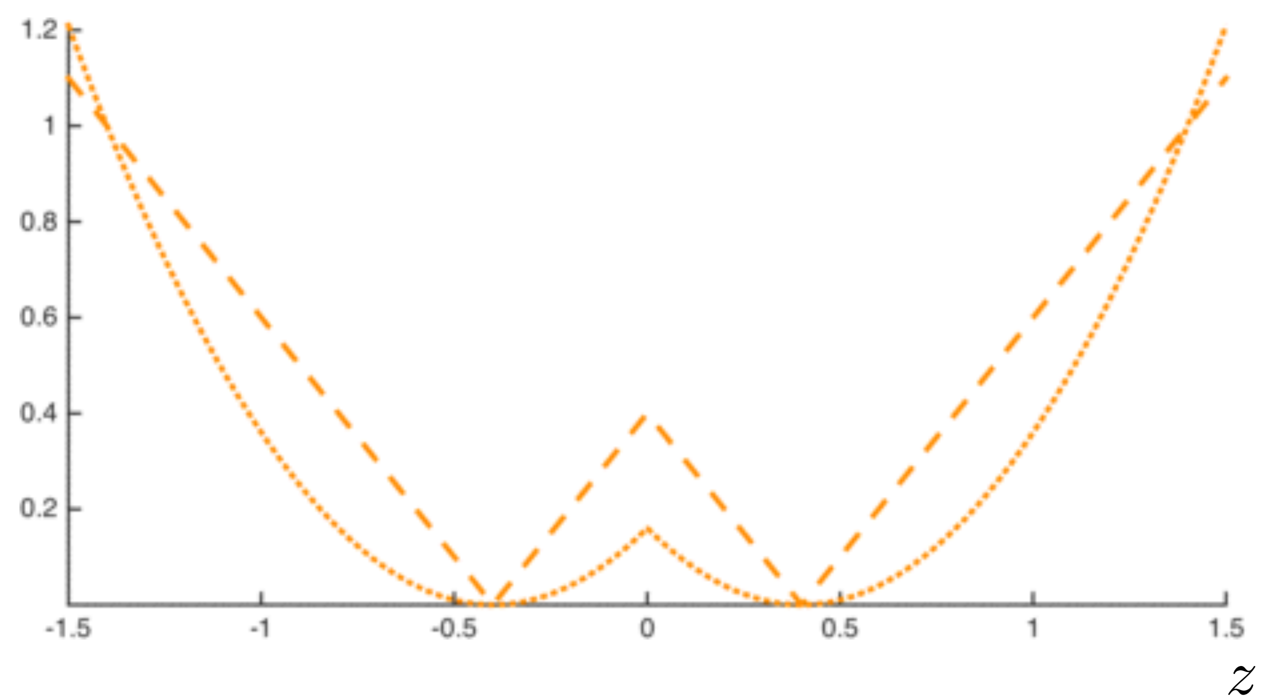
Dissimilarity measures

$f_Q(z) = (\|z\| - d)^2$



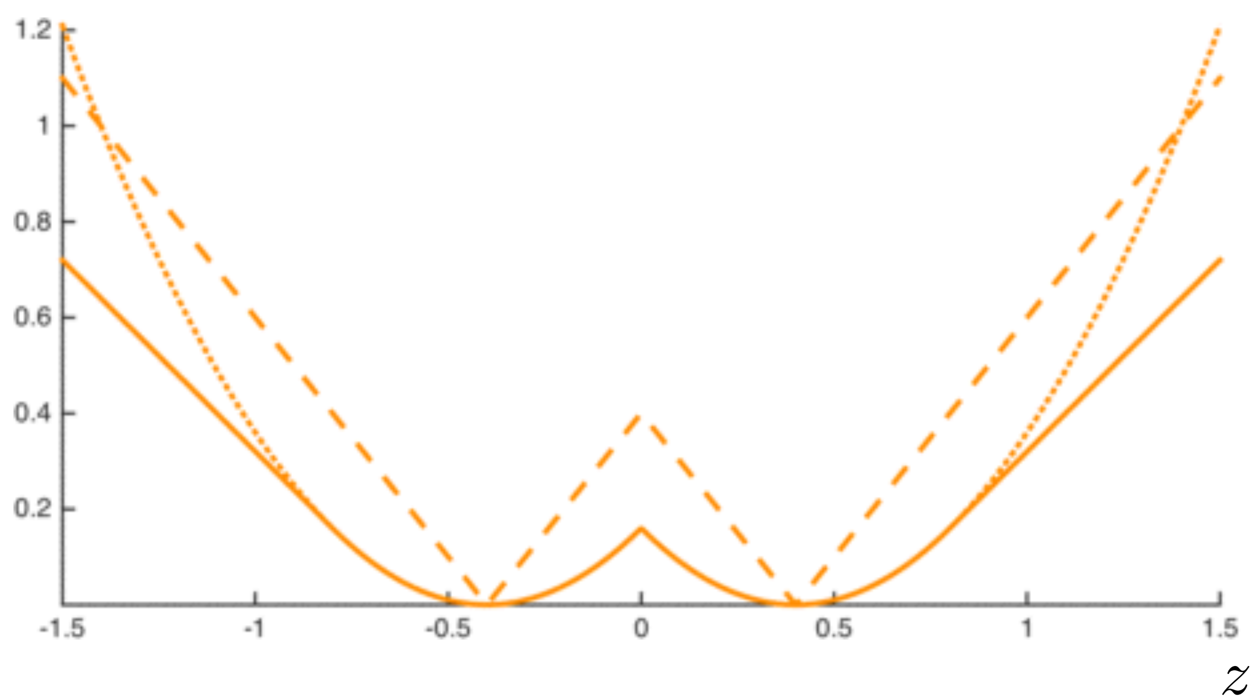
Dissimilarity measures

$f_Q(z) = (\|z\| - d)^2$ $\qquad$ $f_{|\cdot|}(z) = |\|z\| - d|$



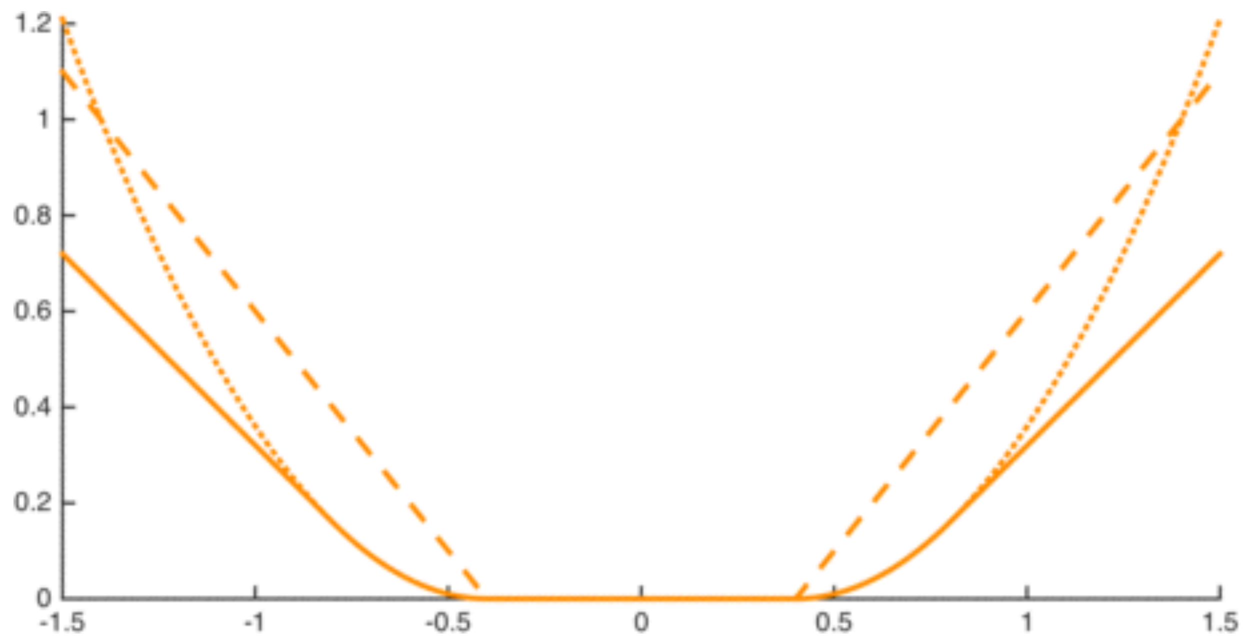
Dissimilarity measures

$f_Q(z) = (\|z\| - d)^2$ $f_{|\cdot|}(z) = |\|z\| - d|$ $f_R(z) = h_R(\|z\| - d)$



Convex underestimators

$$\hat{f}(x) = f\left(\max\left\{0, (\|z\| - d)\right\}\right)$$



Convex underestimators

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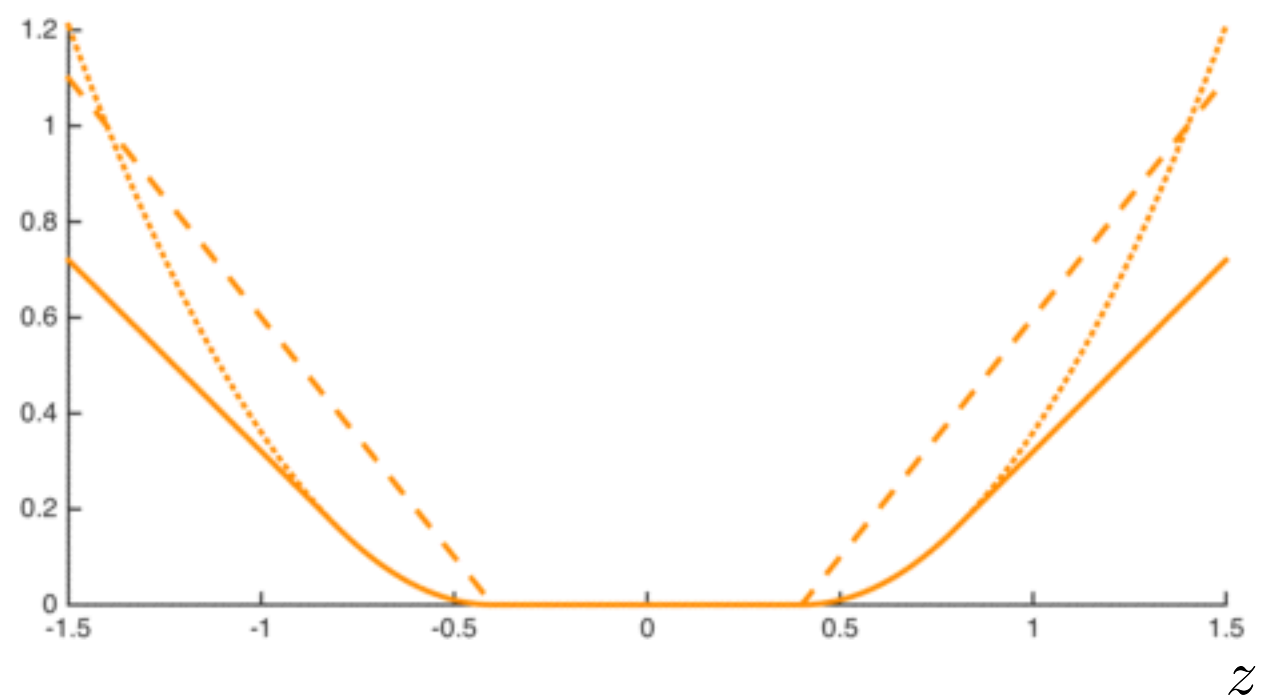
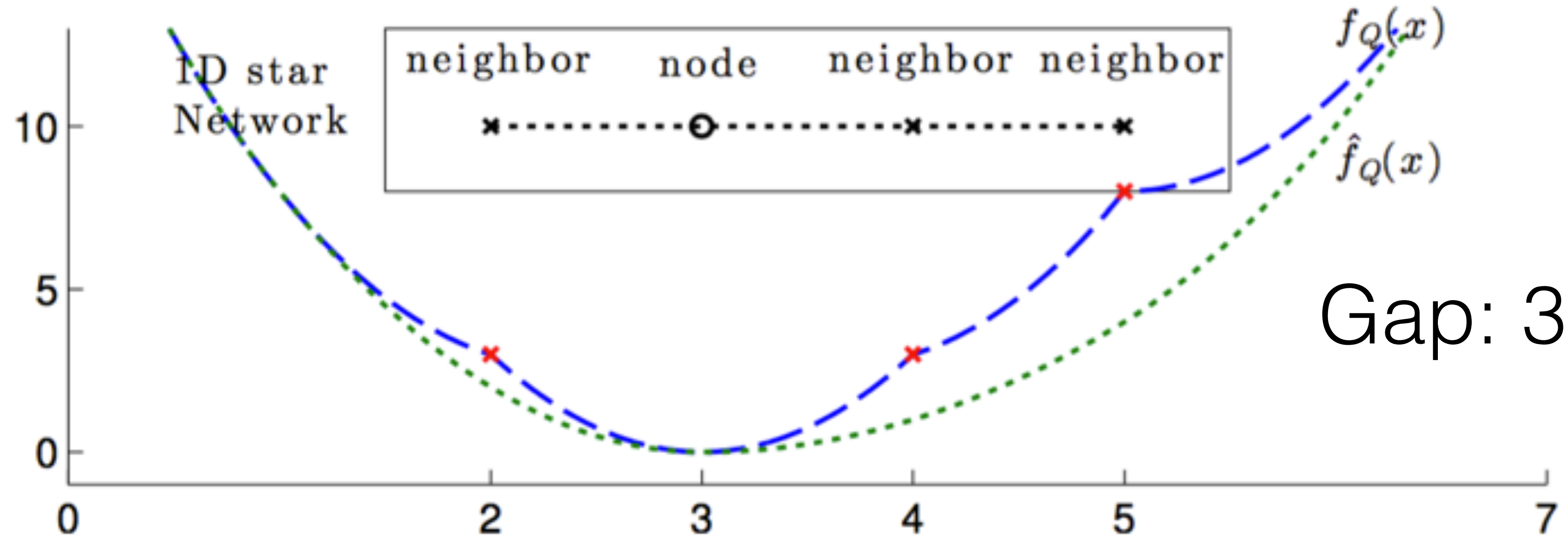
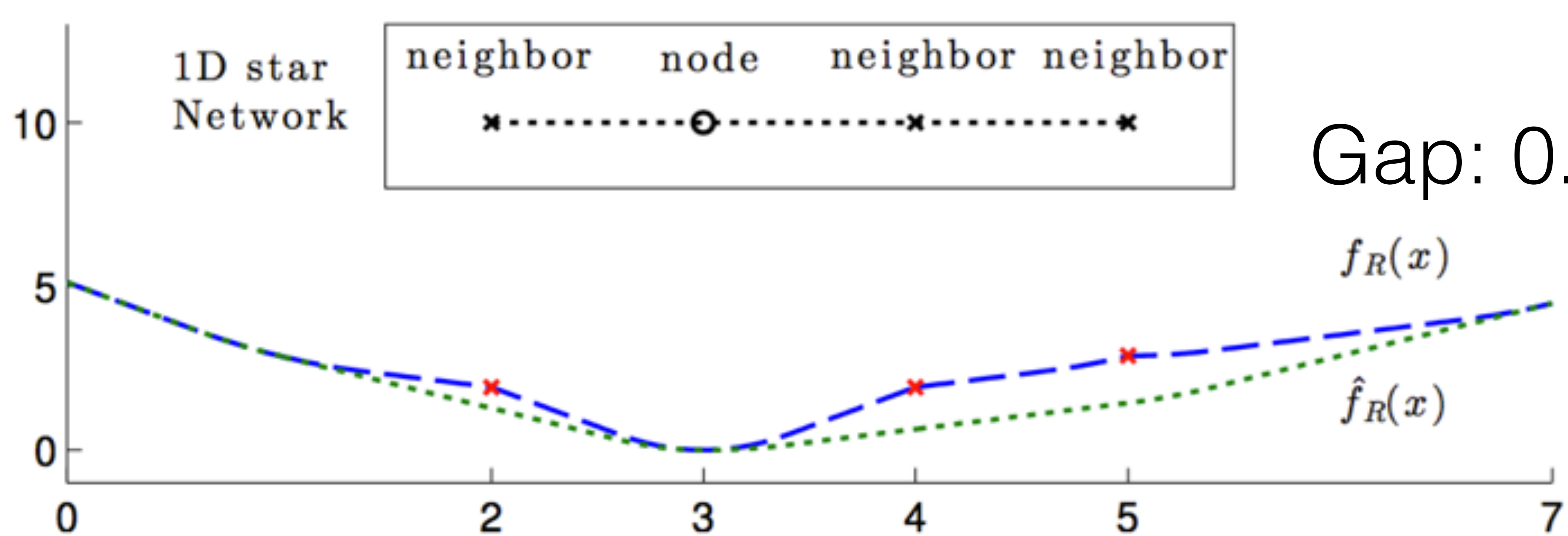


Optimality gap

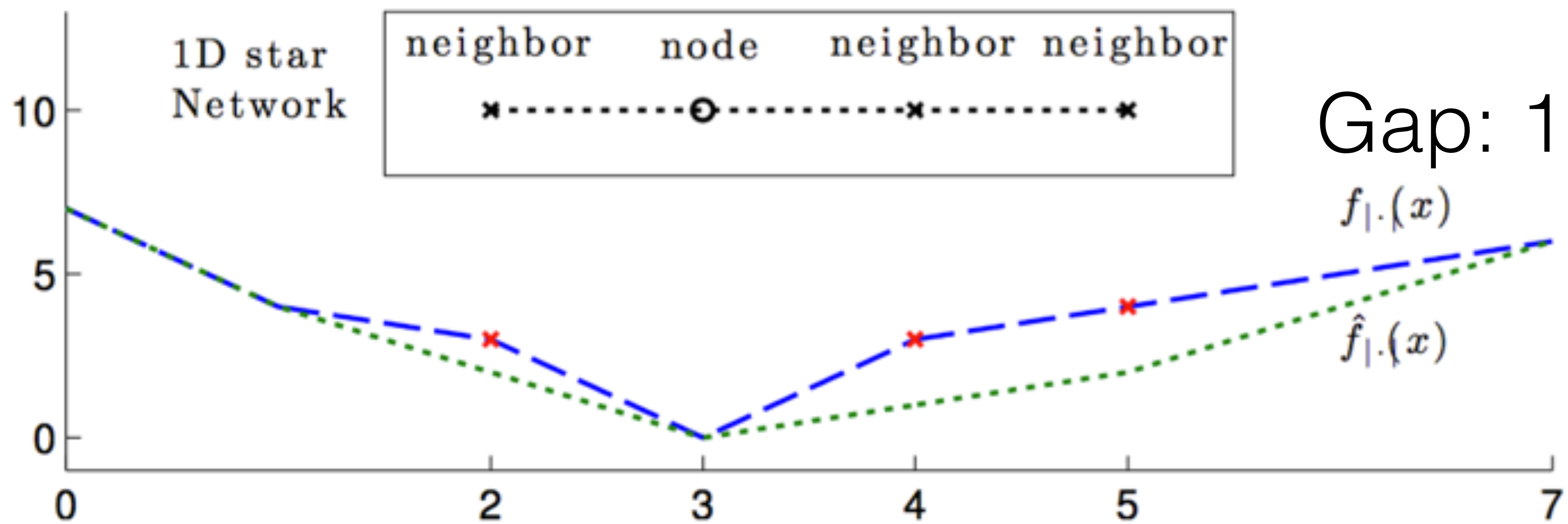


Gap: 3.7019

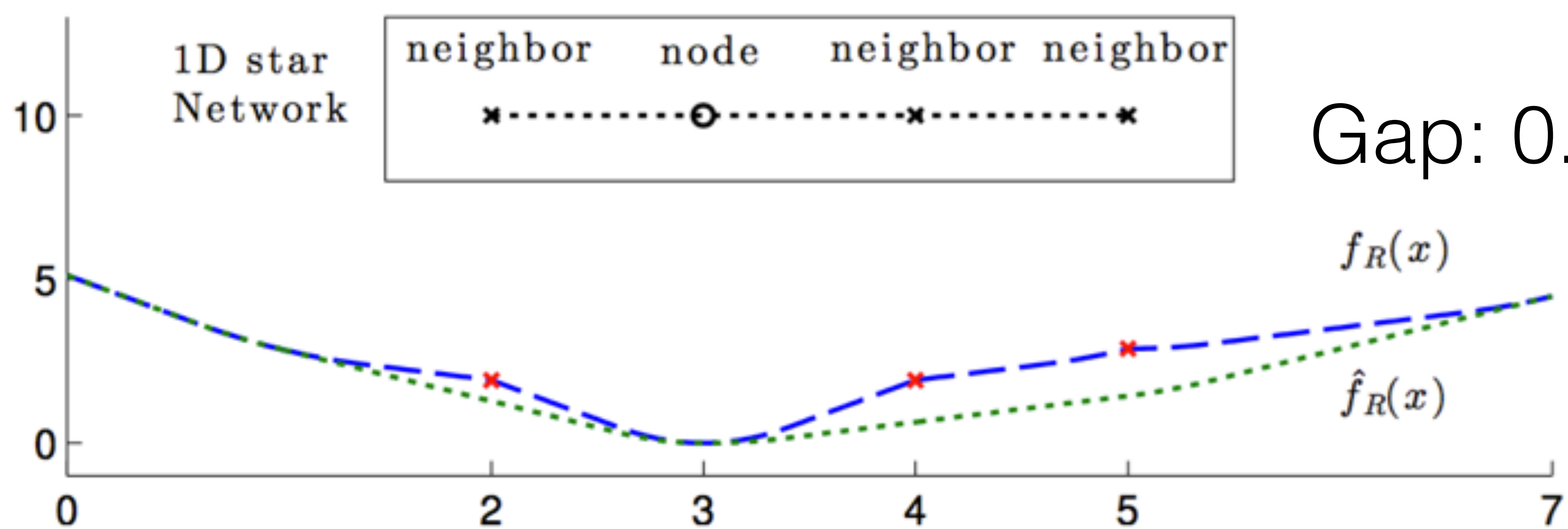


Gap: 0.1784

Optimality gap

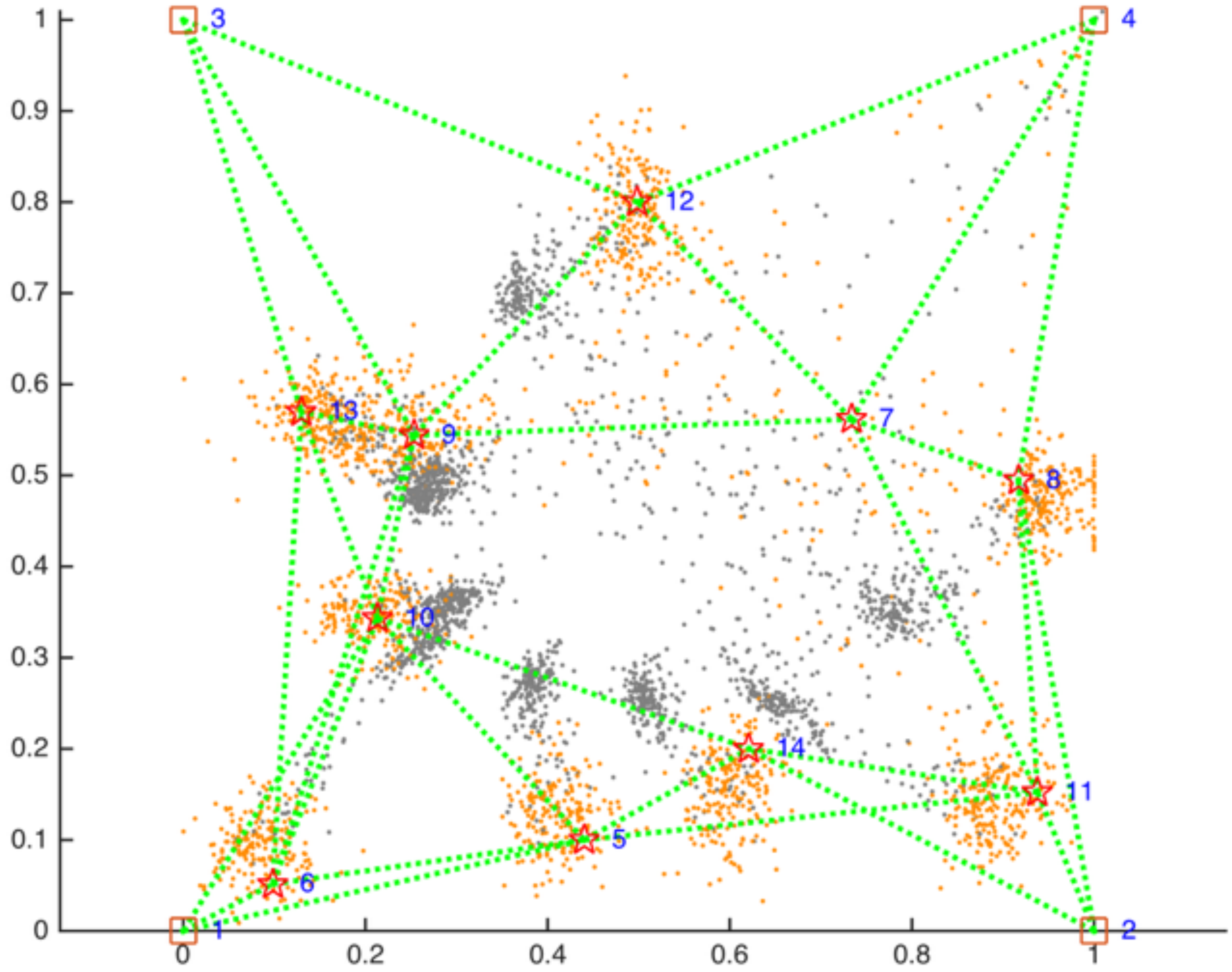


Gap: 1.1416



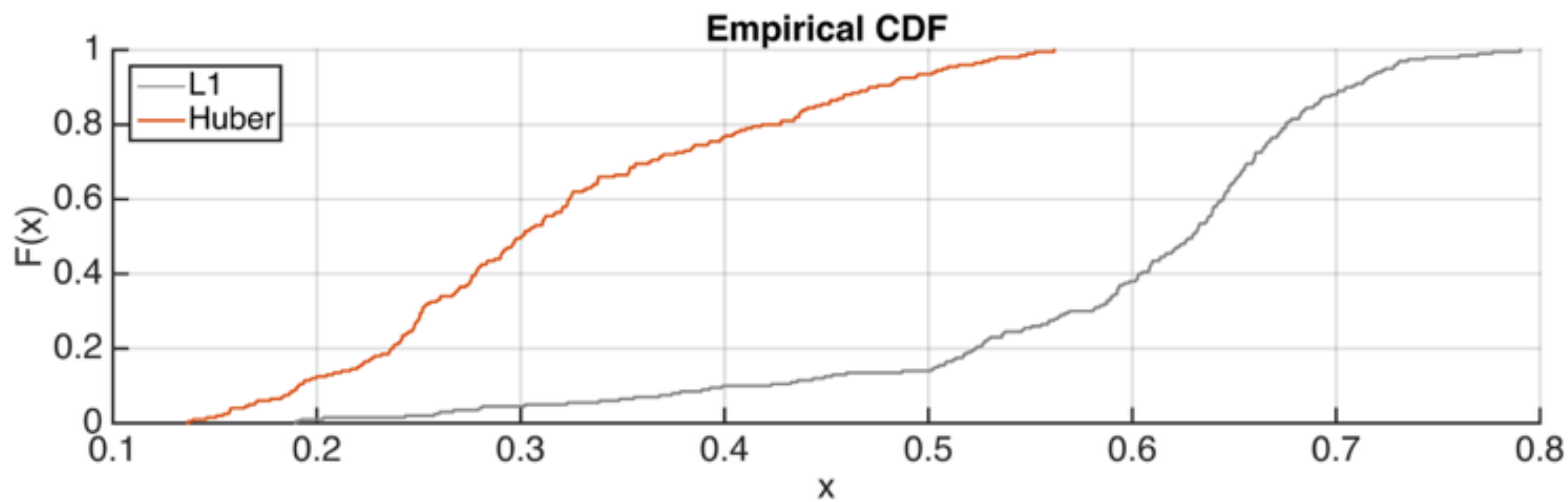
Gap: 0.1784

Experimental results

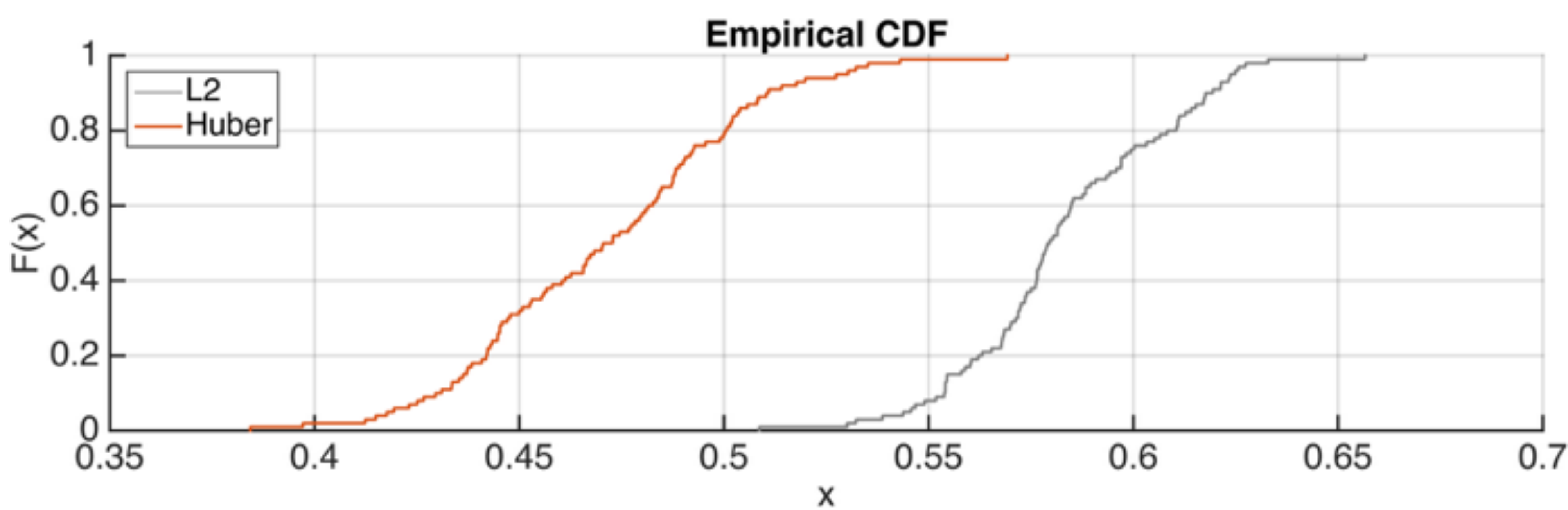


Experimental results

Positioning Error

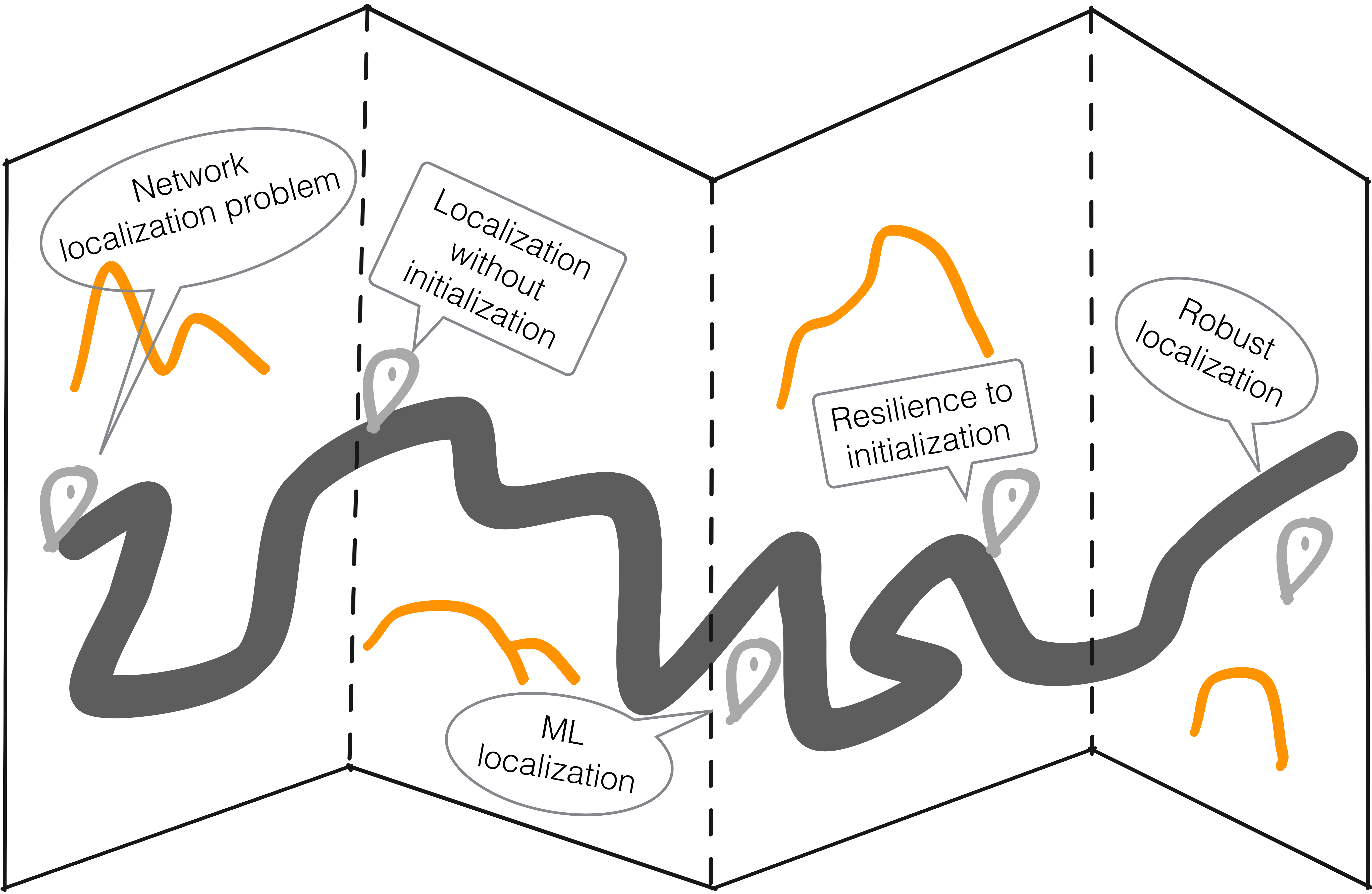


Positioning Error — biased experiment



Main contributions

- General purpose approach: no outlier model is assumed;
- Tightness analysis of the convex underestimator;
- Novel convex formulation resilient to data outliers;
- Distributed synchronous algorithm with optimal conference rate;
- Distributed asynchronous algorithm with proven convergence.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