

Idling Policies for Periodic Review Inventory Control^{*}

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Abstract: This paper presents a study on idling production policies in the context of periodic review inventory control. The usefulness of these policies will be demonstrated by means of two different numerical experiments from which we will be able to extract some interesting managerial insights, as well as some structural properties of the solutions. In the first experiment, the stabilization properties of these policies will be illustrated by means of the stabilization of the Lu&Kumar network. In the second experiment, we will present a stable system for which idling policies achieved a lower operational cost when faced with their non-idling counterparts.

Keywords: Inventory Control, Idling Policies, Infinitesimal Perturbation Analysis, Simulation Based Optimization.

1. INTRODUCTION

In the context of industrial applications, inventory control policies address the problem of splitting a finite amount of capacity among a set of different products that need processing. In the context of periodic review inventory control, the base stock policy is known to be optimal for single product, single machine systems (capacitated or uncapacitated). Also, for uncapacitated machine flow lines it is known that the Multi-Echelon Base Stock policies – MEBS – are optimal, (Clark and Scarf, 1960). However, when machines are capacitated, little is known about the structure of the optimal inventory policy, except that the optimal final product inventory levels are bounded. Despite knowing this, there are no closed form results when it comes to determining the optimal policy nor these bound values. When using MEBS one still has to address how to split capacity among the different products when their requirements are above the capacity value. Many authors have used variations of the MEBS proposing different dynamic rules to address the capacity split. Examples are Priority, Linear Scaling, Equalize Shortfall, and Weighted Equalize Shortfall, among others (See (Janakiraman et al., 2009)). One particular feature of all these variations is the fact that they all are non-idling. The production decisions for a given decision period are solely limited by feeding inventory and machine capacity.

We claim and show that tighter limits to the production decisions may be needed to guarantee stability and to achieve better performances. The main objective of the work described here is to study idling policies for periodic review inventory control and to establish a framework where they can be compared with the non-idling approaches.

This paper presents an example of a stable system where these tighter production bounds are used to achieve performance improvements. The stabilization of an unstable network similar to the one described in Lu and Kumar (1991) using these production bounds to induce idleness will also be presented.

This paper is organized as follows. We start by introducing the theoretical model as well as its dynamic equations, Section 2. The Infinitesimal Perturbation Analysis as well as its applicability will be discussed in Section 3. Section 4 will present the stabilization of a network similar to the one of Lu&Kumar using the proposed policies. In Section 5 we will present an experiment where, using IMEBS, we were able to improve operational cost over their non-idling counterparts. The paper ends, in Section 6, with some conclusions and references to future work.

2. THEORETICAL MODEL

In (Bispo, 1997), the author presents a framework to study MEBS, in the context of periodic review inventory control, for simple re-entrant systems producing multiple products.

The framework here presented is an extension of the one introduced in (Bispo, 1997), which will now contemplate non-acyclic layouts, as well as a set of production bounds capable of inducing idleness in the production policies. These bounds will directly impact equation (6) below. For a more detailed description of the framework, we refer the reader to (Santos, 2016), of which this paper constitutes a synthesis.

2.1 Framework

Consider an eventually non-acyclic, multi-stage, multi-product, capacitated production system facing random

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demand, being a capacitated production system, a system whose machines have a limited amount of capacity available at any point of time. Each one of the P products follows a specific routing pattern defined by a set O^p of operations. Every set of operations O^p is indexed from K to 1 being K the first operation (most upstream) and 1 the last. When analysing a given operation, the set O^p will always have the information of which are its upstream and downstream operations. This information is what makes possible the generalization of the model to non-acyclic layouts.

2.2 Basic recursions

Inventory Dynamic Equation For the sake of simplicity, in the following set of equations, I_n^{kp} and P_n^{kp} refer, respectively, to the inventory and production values of operation $O^p(k)$ at the decision period n respectively. d_n^p represents the demand value for product p at the decision period n .

$$I_{n+1}^{kp} = \begin{cases} I_n^{1p} - d_n^p + P_n^{1p} & k = 1, \\ I_n^{kp} - P_n^{(k)-p} + P_n^{kp} & \text{otherwise.} \end{cases} \quad (1)$$

Equation (1) describes the evolution of the inventory levels throughout the operation. The inventory level for the most downstream operation (I^{1p}), decreases by the amount of product p which leaves the system by means of the demand process and increases by the amount produced at this stage. The other production levels will see their inventory levels decrease by the amount of product that their downstream operations consume.

Note that $P_n^{(k)-p}$ represents the amount of product p produced at decision period n for operation $O^p(k-1)$.

Echelon Inventory Dynamic Equation The echelon inventory of $O^p(k)$ at decision period n is defined as:

$$E_n^{kp} = \sum_{x=1}^k I_n^{xp}. \quad (2)$$

Locally, a stage has usually no visibility over the downstream inventory levels and, for that matter, equation (2) is not applicable. The dynamic evolution of the echelon inventory may be described by means of an alternative equation:

$$E_{n+1}^{kp} = E_n^{kp} - d_n^p + P_n^{kp}. \quad (3)$$

Analysing the previous equation, the echelon inventory will grow with the amount of production at that corresponding stage and will decrease with the amount of products leaving the system.

Shortfall Dynamic Equation The shortfall is defined as the difference between the echelon base stock and the echelon inventory:

$$Y_n^{kp} = z^{kp} - E_n^{kp}. \quad (4)$$

It is possible to write a dynamic equation for the shortfall similar to the one for echelon inventory given by:

$$Y_{n+1}^{kp} = Y_n^{kp} + d_n^p - P_n^{kp}. \quad (5)$$

Production net needs Production net needs represent the production quantities that the system needs when there are no capacity bounds. In order to ensure that the capacity of the machines is not exceeded, production rules will be applied to the production needs in a posterior step. The production net needs may be bounded by upstream inventory or by the forced bounds imposed to the system. Let us define \bar{I}^{kp} the production bound imposed to product p at operation k . When there are no bounds, the system will always try to produce the sufficient amount to take the shortfall of the buffers to zero. The production net needs for a given product and operation are defined by:

$$f_n^{kp} = \begin{cases} \min \{ (Y_n^{Kp} + d_n^p)^+, \bar{I}^{Kp} \} & k = K, \\ \min \{ (Y_n^{kp} + d_n^p)^+, I_n^{(k)+p}, \bar{I}^{kp} \} & \text{otherwise.} \end{cases} \quad (6)$$

where $(x)^+ = \max \{0, x\}$. Note that the first machine of a production line will never be limited by inventory, since raw material is assumed to be always available.

The echelon base stock variables must respect the following rule $z^{kp} \geq z^{(k)-p}$. As discussed in Bispo (1997), instead of continuously compare two consecutive multi-echelon base stock levels throughout the optimization procedure, it is preferable to use an alternative set of variables where this rule is simplified.

$$\Delta^{kp} = \begin{cases} z^{kp} & k = 1 \\ z^{kp} - z^{(k)-p} & \text{otherwise.} \end{cases}$$

During the optimization procedure it is much easier to enforce that every $\Delta^{kp} \geq 0$ instead of making sure that the base stock variables are always non increasingly ordered.

Priority Rule Assuming that, every machine $m = 1, \dots, M$ has a set of operations J^m and that $J^m(x)$, for $x = 1, \dots, X^m$, is the operation that comes in the x th position on the priority list. That is, $J^m(1)$ is the operation with the highest priority and $J^m(X)$ has the lowest priority. The production decision for machine m will be:

$$\begin{aligned} P_n^{J^m(1)} &= \min \left\{ f_n^{J^m(1)}, \frac{C^m}{\tau^{J^m(1)}} \right\}, \\ &\vdots \\ P_n^{J^m(x)} &= \min \left\{ f_n^{J^m(x)}, \frac{C^m - \sum_{j=1}^{x-1} \tau^{J^m(j)} P_n^{J^m(j)}}{\tau^{J^m(x)}} \right\}. \end{aligned} \quad (7)$$

Note: $\tau^{J^m(x)}$ is the capacity required to produce one product unit on operation $J^m(x)$ in the machine m .

Linear Scaling Rule Product decision rule for operation x of machine m :

$$P_n^{J^m(x)} = f_n^{J^m(x)} g_n^m, \quad (8)$$

$$g_n^m = \min \left\{ \frac{C^m}{\sum_{j=0}^X \tau^{J^m(j)} f_n^{J^m(j)}}, 1 \right\}, \quad (9)$$

Operational Cost The operational cost refers to the cost of stocking inventory and being penalized by backlogs. Let a single stage cost be defined as

4. PROXY OF THE LU&KUMAR SYSTEM

In (Lu and Kumar, 1991), in the context of Queuing Networks, the authors show that not all priority policies are stable. In fact, the authors present an example where a specific choice of priorities will induce instability in a system.

As stated in (Bispo, 1997), IPA is not valid for re-entrant systems working under the priority rule when the values of τ are not all equal. For this reason, the classical Lu&Kumar system cannot be studied with our model.

In (Dai and Weiss, 1996), the Lu&Kumar system is widely studied in the context of fluid models. In the mentioned paper, the authors propose a proxy layout for the Lu&Kumar system which, under the same conditions, is unstable, and it is not a re-entrant system, see Figure 1. The numerical study here presented was conducted using this proxy layout.

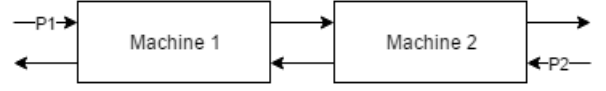


Fig. 1. Lu&Kumar proxy layout.

In this layout, at machine 1 priority is given to operation $O^1(2)$ over $O^2(1)$, while at machine 2 priority is given to operation $O^2(1)$. The mean demand rate considered is $8/7$ while each machine has a total capacity of 10 (80% load). The τ values considered in this example are given in Table 1.

Table 1. τ values structure.

	τ^{11}	τ^{21}	τ^{12}	τ^{22}
Value	1	6	1	6

Figure 2, displays the inventory paths for the system of (Dai and Weiss, 1996) under the mentioned conditions, when using MEBSP.

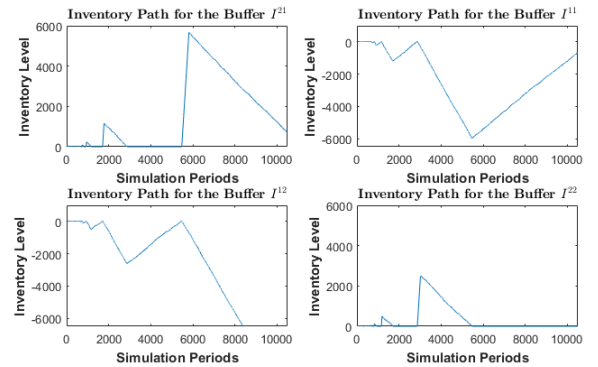


Fig. 2. Inventory paths for the Lu&Kumar proxy system.

The presented behaviour is similar to the one of Lu&Kumar as one can observe that the system inventory buffers present the same cyclic growth as the ones of the previous example.

$$C_n = \sum_{p=1}^P C_n^p, \quad (10)$$

where C_n^p is given by

$$C_n^p = (I_n^{1p})^- b^p + (I_n^{1p})^+ h^{1p} + \sum_{k=2}^{K^p} I^{kp} h^{kp}, \quad (11)$$

and K^p represents the number of operations from the set O^p . b^p and h^{kp} are the standard backlogging and holding costs.

With the single period cost established in 10, one is now capable of calculating the finite horizon average operational cost taking into account N simulated periods:

$$C_N = \frac{1}{N} \sum_{n=1}^N C_n. \quad (12)$$

Initial conditions The state variables will be set to their base stock variables, that is $I_0^{kp} = \Delta^{kp}$. The echelon inventories will be set according to $E_0^{kp} = \sum_{i=1}^k \Delta^{ip}$. The forced bounds \bar{I}^{kp} will be set to a value in the interval between the machine load and its total capacity. All other initial variables are set to zero.

3. INFINITESIMAL PERTURBATION ANALYSIS

To show validity of IPA it is necessary to demonstrate that the expected value and derivative are interchangeable operators. If one takes into account the finite horizon average cost equation (12), if the system is simulated during an amount of decision periods long enough, for i.i.d. demands, so that it can cover in time the same amount information that it would cover for all sample paths, the following approximation is valid due to the Law of Large Numbers:

$$C_N = \frac{1}{N} \sum_{n=1}^N C_n \rightarrow \bar{C}_N = \frac{1}{N} \sum_{n=1}^N E[C_n] \text{ for } N \gg, \quad (13)$$

where $E[\cdot]$ is the expected value over all sample paths.

The main result concerns the cost derivatives. When IPA is valid, the following relationship holds:

$$\frac{\partial}{\partial \theta} \bar{C}_N = \frac{1}{N} \sum_{n=1}^N \frac{\partial}{\partial \theta} E[C_n] = \frac{1}{N} \sum_{n=1}^N E\left[\frac{\partial}{\partial \theta} C_n\right]. \quad (14)$$

For the same reasons as (13), it holds that:

$$\frac{1}{N} \sum_{n=1}^N \frac{\partial}{\partial \theta} C_n \rightarrow \frac{1}{N} \sum_{n=1}^N E\left[\frac{\partial}{\partial \theta} C_n\right] \text{ for } N \gg. \quad (15)$$

Therefore, we can get estimates of the derivatives by simply using one long enough sample path. The consequence being that the simulation based optimization is more efficient, given that one single simulation suffices to obtain gradient estimates, (Glasserman, 1991). The left side of (15) is obtained by direct derivation of the equations presented in section 2. Resorting to a single sample path is also advantageous from the variance reduction stand point, as discussed in L'Ecuyer (1994).

Dai and Weiss (1996) propose a stability condition for this system which can be translated to the inventory control paradigm notation:

Remark 1. Assume that

$$\frac{E[d^1]\tau^{21} + E[d^2]\tau^{12}}{C} < 1 \text{ and } \frac{E[d^1]\tau^{11} + E[d^2]\tau^{22}}{C} < 1.$$

There exists an unstable policy for the model if and only if

$$\frac{E[d^1]\tau^{21} + E[d^2]\tau^{22}}{C} \geq 1$$

where C is the machine capacity.

In the example of Figure 2, as we can see in Table 1, the values of τ^{21} and τ^{22} along with the expected demand rate for both products are in violation of the stability condition.

With the application of IMEBSP to the system, we were in fact capable of achieving stability as we show in Figure 3.

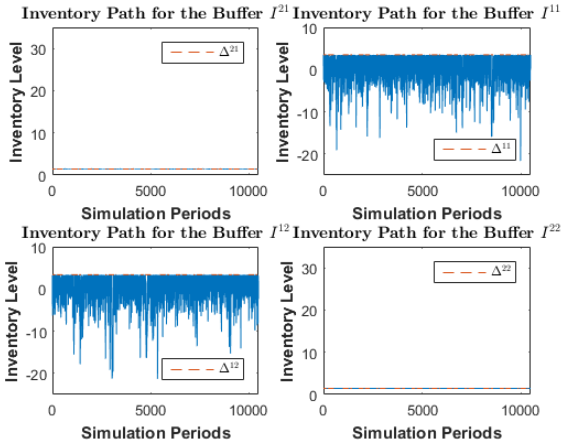


Fig. 3. Inventory paths for the stabilized proxy system.

Under IMEBSP, the inventory buffers are able to regenerate to their initial conditions multiple times throughout the simulation, with finite expected time between regeneration points. Also, the operation $O^2(2)$ is now working under a local base stock policy. We must stress that the system setup is the same as the presented in the previous unstable examples.

Table 2. Optimal production bound parameters.

	\bar{I}^{11}	\bar{I}^{21}	\bar{I}^{12}	\bar{I}^{22}
Production Bound Value	1.43	1.42	1.43	1.43

Analysing the optimal bound parameters registered in Table 2, we can extract two very interesting conclusions. The first being the fact that the production bounds are in fact making a weighted division of the machines capacities among their operations using the τ parameters as weights:

$$\frac{\tau^{11} * \bar{I}^{11} + \tau^{21} * \bar{I}^{21}}{C} = \frac{1 * 1.42 + 6 * 1.43}{10} = 1.00,$$

$$\frac{\tau^{12} * \bar{I}^{12} + \tau^{22} * \bar{I}^{22}}{C} = \frac{1 * 1.43 + 6 * 1.43}{10} = 1.01.$$

The second conclusion comes from the fact that the optimal solution, under an idling policy, manages to stabilize the system working in a region where (Dai and Weiss, 1996) does not guarantee stability for non-idling policies. Given that the Lu&Kumar system respects the traffic condition, it is stable under IMEBSP.

Table 3. Percent of decision periods where idleness occurs.

	IMEBSP	MEBSP
Machine 1	61.23%	60.07%
Machine 2	100.00%	31.96%

As we can observe from the percent of periods where idleness occurs under each policy presented in Table 3, the IMEBSP resorts to the inclusion of additional idling periods in order to induce stability. In fact, under IMEBSP, it will be optimal to never let machine 2 to use its full capacity.

5. THE CRISS CROSS SYSTEM

The Criss Cross system has been widely studied in the queuing networks literature (See (Harrison and Wein, 1989) and (Budhiraja et al., 2005) for some examples of these studies). This is a system where the optimal scheduling policy may be idling.

The Criss Cross layout consists of a system with two machines and two product lines, as shown in Figure 4. The system will be tested using two different production rules at machine 1:

- Linear Scaling Rule - LSR
- Priority Rule - PR

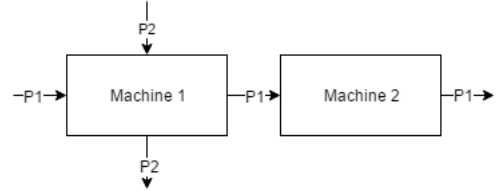


Fig. 4. Criss Cross manufacturing layout.

Note that the priority rule will be tested twice, giving priority to each of the products on machine 1. In order to clarify the used notation, the priorities will be identified in the following manner:

- P_{12} - Product 1 has priority over Product 2
- P_{21} - Product 2 has priority over Product 1

This experiment used the holding cost structure presented in Table 4, while the backlog penalty costs were set to 20.

Table 4. Holding cost structure.

Holding Costs	h^{11}	h^{21}	h^{12}
Cost	10	4	10

Machine 1 has an overall load of 80% and the load of machine 2 varies between 30% and 80%. Moreover, the load imposed by each product in machine 1 is 40% (a 50% load split). The mean demand values are fixed to 10 and the coefficients of variance are equal to 1.

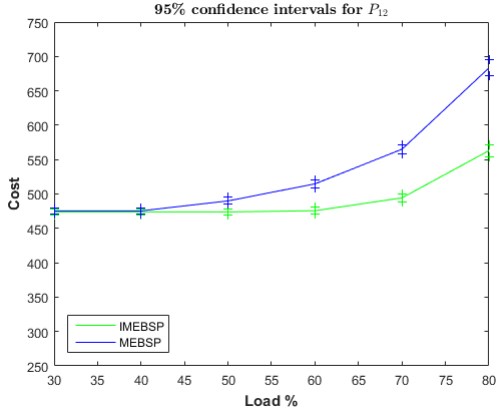


Fig. 5. Mean cost evolution of the Criss Cross system - PR: P_{12} .

In Figure 5 we present the optimal average operational cost as a function of the second machine load for both IMEBSP and MEBS. This experiment was conducted using P_{12} . Observing the mean cost evolution presented in Figure 5, one can conclude that, for high loads, the IMEBSP is able to achieve close to a 19% cost reduction margin when compared to MEBS.

The figure also shows the 95% confidence intervals of the obtained results. It is important to state that these confidence intervals, given their size, show that the simulation length is long enough, ascertaining the validity of equation (13) and (15).

Observing Figure 6, we can conclude that product 2 does not need to be bounded since the optimal value of its bound converges to the same value as the full capacity of machine 1. This means that, at any given period, product 2 will be allowed to use all of this machine capacity.

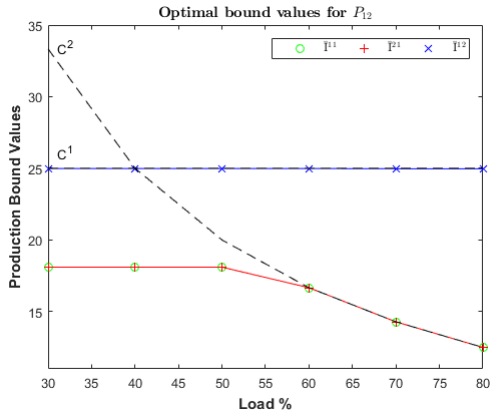


Fig. 6. Optimal production bound levels for the Criss Cross system - PR: P_{12} .

Analysing the bound applied to machine 2 there are two different observations to make. Before this machine reaches the mark of 60% load, the value of its bound is always placed under the value of both machines total capacity. This means that, under these conditions, this machine may be forced into idleness in some decision periods. When the load of machine 2 is raised above 60%, the bound applied to this machine converges to its full capacity and, in this

case, the production at this level will only be limited by the upstream inventory and/or shortfall.

The bound applied to the first operation of product 1 shows a very interesting behaviour. When the load of machine 2 is low, machine 1 is the bottleneck for product 1. The optimal value of the \bar{I}^{21} will nonetheless always be below machine 1 capacity. This bound will induce an alteration of machine 1 strict priorities. In machine 1, working under the IMEBSP, product 1 only has priority until the bound value is produced. All the capacity that is left will be available for product 2 and/or wasted. This phenomena will act like as a “relaxation” of the priority rule that will never let the high priority product monopolise the machine capacity.

When machine 2 has high load, it becomes the bottleneck for product 1. In this case, \bar{I}^{21} will converge to the value of the second machine capacity. This means that, for a given period, the first machine will never produce an amount of product 1 that is above the value of the bottleneck capacity.

Figure 7 shows the inventory tracking for the optimized system working at 80% load on both machines as well as the optimal Δ values for which the system converged.

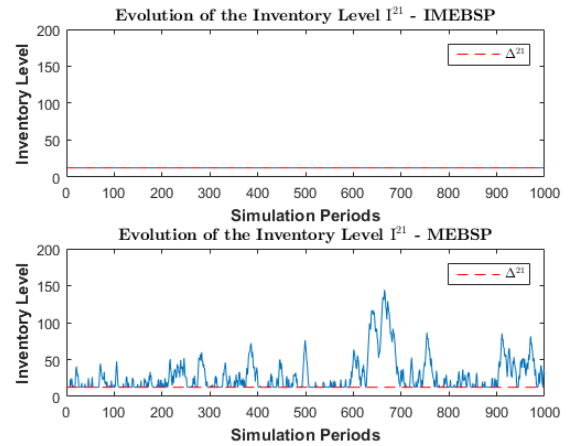


Fig. 7. Inventory path of I^{21} .

The bound value in machine 1 completely changes the way inventory is handled at the I^{21} . Given that the \bar{I}^{21} is set to the same value as machine 2 capacity and that Δ^{21} converges to the same value, this buffer will have zero variance on the stock level at all times.

At any decision period, machine 2 will consume a maximum value of the buffer I^{21} which is equal to its capacity. Given that both the optimal \bar{I}^{21} and the optimal Δ^{21} are set to the same value as machine 2 capacity.

Table 5. Registered variance at the inventory buffers.

Inventory Variance	IMEBSP	MEBSP
I^{11}	28,17	28,17
I^{21}	0	24,63
I^{12}	19,66	24,79

Table 5 presents the variance of the inventory levels throughout the simulation. As a reduction in the operational costs is directly correlated with a reduction of

the overall variance level at the buffers, this result serves as a justification on how the IMEBSP achieves a better performance when faced with its non-idling counterpart.

At this point one question remains to be answered: how much idleness are the bounds forcing? This information will help us conclude if the IMEBSP is in fact behaving as a non-idling policy or if the application of the bounds is simply regulating the priority rule. Table 6 registers the percent of decision periods where capacity was not fully used, while both machines worked at 80% load.

Table 6. Percent of decisions where the machines do not use their full capacity.

Idle Periods	IMEBSP	MEBSP
Machine 1	54.10%	44.84%
Machine 2	36.66%	38.19%

As noted before, when both machines are working at 80% load, the only bound having an impact in the system operation is \bar{I}^{21} which is applied in the first machine. Analysing Table 6 one can conclude that the idling policy results in a substantial increase of the amount of periods that machine 1 does not use all of its capacity.

On the other hand, the second machine registered a smaller amount of periods where capacity was not fully used. This behaviour may be justified by the reduction of the stock variance at this machine entrance. Given that this machine will always have an amount of stock at its entrance that is equal to its capacity, its production will never be limited by the upstream stock level and, for that matter, it will run in a much smoother manner. What Table 6 shows is that the extra idleness incurred by machine 1 benefits machine 2, which is more utilized, and consequently the whole system gains.

Table 7 presents the comparison of the results achieved by the three different decision methods when both machines were working at 80% load.

Table 7. Cost benchmark for high load operation.

Rule	IMEBSP Cost	MEBSP Cost	Cost Improvement
P_{12}	577.97	707.95	18.4%
P_{21}	625.23	633.73	1.5%
LSR	590.61	600.44	1.6%

When it comes to the evaluation of these results it is important to notice that all the production rules showed an overall cost improvement while under an idling policy. Nevertheless, it should be pointed that the priority rule P_{12} , which was the one with the worst performance in the MEBSP policy, was able to beat both the LSR and the Priority rule P_{21} in IMEBSP.

6. CONCLUSIONS

We presented two distinct systems, which have been widely studied in the context of Queuing Networks, and addressed them in the context of Periodic Review Inventory Control. While it was not expected that a change in model paradigm would modify these systems behavior, we argue that the framework of Periodic Review Inventory Control is more amenable to the definition and usage of idling policies.

We proposed a new class of policies, the IMEBSP, which are a generalization of the classical MEBSP, and presented some numerical experiments showing two main features of these policies:

- The performance of IMEBSP was never worse than the performance of the MEBSP;
- IMEBSP is able to stabilize systems that may be unstable under their MEBSP counterparts.

Under MEBSP there are only two situations that may prevent a machine to use its full capacity: shortfall is below machine capacity, feeding inventory is insufficient. By using IMEBSP we add a third situation that prevents the full capacity to be used: the production bounds. Whereas the first two situations can be considered as passive idleness, this third situation is clearly active idleness. That is, active idleness is a choice of the decision maker while passive idleness is simply a consequence of the demand path. The numerical results show that the possibility of choosing idling periods results into a more balanced utilization of all resources and an overall reduction on the inventory buffer variances, in the sense that it works as filter on the burstiness of the demand process. It is well known that performance costs are negatively influenced by the variance of the inventories.

The validation of the IPA approach was out of the scope of the present paper. This is an issue that will have to be addressed in the future.

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